# A Topological Method for Finding Invariant Sets of Switched Systems


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## ABSTRACT

We revisit the problem of finding controlled invariants sets (viability), for a class of differential inclusions, using topological methods based on Ważewski property. In many ways, this generalizes the Viability Theorem approach, which is itself a generalization of the Lyapunov function approach for systems described by ordinary differential equations. We give a computable criterion based on SoS methods for a class of differential inclusions to have a non-empty viability kernel within some given region. We use this method to prove the existence of (controlled) invariant sets of switched systems inside a region described by a polynomial template, both with time-dependent switching and with state-based switching through a finite set of hypersurfaces. A Matlab implementation allows us to demonstrate its use.



## Categories and Subject Descriptors

G.1.7 [**Mathematics of Computing**]: Numerical Analysis; F.1.1 [**Theory of Computation**]: Computation by Abstract Devices

## Keywords

control, differential inclusion, viability, cyber-physical systems


## 1. INTRODUCTION

In order to understand and control the dynamics of systems ruled by differential equations, it is important to locate regions of the phase space that contain "invariant sets", i.e., sets of points that are invariant under the action of the dynamical system. Topological methods, based on Ważewski property and Conley index [20], have been used with success in order to find such invariant sets, within prescribed semi-algebraic sets (or "templates") and give their qualitative behavior: periodic orbits, attractors, repellers, chaos, ... (see, e.g., [21]). The boundary of these templates are decomposed into "exit sets" and "entrance sets" according to the directions of the flow at these points. It is well-known, for example, that if all the flows are either entering into or exiting from the template, then there exists an invariant inside the template. The Ważewski property, in particular, gives criteria for guaranteeing the existence of invariants, in more general cases with both entering and exiting flows at the boundary. This method has been used in our previous work on continuous systems [8]. We extend here this work by considering the case of *switched systems*. Switched systems are dynamical hybrid systems that combine continuous and discrete dynamics. These systems are more and more used in industrial applications, such as power electronics, due to their versatility and ease of implementation. A switched system is defined by a family of continuous dynamics, and by a switching signal that changes the operating mode of the system from one dynamics of the family to another. We use here topological methods (in the sense of topological dynamics : we are concentrating on closedness and non-connectedness of the exit set) in order to guarantee the presence of invariants inside templates for different classes of switching signal: time-dependent or state-dependent.




*HSCC'16, April 12 - 14, 2016, Vienna, Austria*





ISBN 978-1-4503-3955-1/16/04...$15.00

DOI: http://dx.doi.org/10.1145/2883817.2883822


### *Related work.*

Similar problems have been investigated in the literature. For example, in [2, 19, 24, 15, 14], authors calculate a controlled invariant set contained in $K$ via an *iterative algorithm*. The algorithm is initialized with $K$ and iteratively removes trajectories that may be forced to exit the set due to system dynamics. If the algorithm terminates at a fixed point, this final set is the maximal controlled invariant set (MCIS) contained in $K$. In general, the algorithm does not terminate, and only an approximation of the MCIS is found, which is still invariant but not maximal.

An alternative approach is to synthesize guards or tuning the parameter values of switching surfaces in order to minimize an integral cost function (see, e.g. [4]). In [12, 10], the authors create a transition system from the hybrid dynamics by partitioning the state-space and introducing transitions between partitions which reflect the dynamics and invariance properties of the hybrid system model. The relation between the hybrid system and the new transition system is called a *bisimulation*, and a controller for the original system can be synthesized from this bisimulation.

In [6], the authors use sum of squares (SOS) programming to synthesize switching laws that are guaranteed to satisfy a state-based safety constraint. They consider hybrid systems with a finite number of modes in which the state evolution is governed by a differential inclusion, and they synthesize guards that trigger transitions between modes. Guards are assumed to be semialgebraic sets, i.e. a guard is a subset of the continuous state space which satisfies a collection of polynomial inequalities and equalities.

Although all these works pursue an objective similar to ours, they use techniques such as Lyapunov function calculation, bisimulation, fix-point iteration, which differ from our topological approach. Besides they generally treat only linear or affine modes while our method is suitable to switched systems with *polynomial* modes. Note also that, as mentioned above, our method is based on the Ważewski property which generalizes the Lyapunov approach, and can treat examples that would have been very difficult to obtain with Lyapunov functions (see [8] for examples).

# 2. DIFFERENTIAL INCLUSIONS AND VIABILITY

## 2.1 Basic facts

Consider the general differential inclusion

$$\dot{x} \in F(x) \tag{1}$$

where $F$ is a map from $\mathbb{R}^n$ to $\wp(R^n)$, the set of subsets of $\mathbb{R}^n$. A function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is a *solution* of Equation (1) if $x$ is an absolutely continuous function and satisfies for almost all $t \in \mathbb{R}$, $\dot{x}(t) \in F(x(t))$ (see [3]). In general, there can be many solutions to a differential inclusion. Throughout the paper we note $S_F(x_0)$ the set of all (absolutely continuous) solutions to the Equation (1).

Definition 1. *[3] The set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a* Marchaud map *if $F$ is upper semicontinuous (in short: u.s.c.) with compact convex values and linear growth (that is, there is a constant $c > 0$ such that $|F(x)| := sup\{|y| \mid y \in F(x)\} \leq c(1 + |x|)$, for every $x$).*

We know from [3] that when $F$ is a Marchaud map, then the inclusion (1) has a solution such that $x(t_0) = x_0$ (for all $x_0$) and for a sufficiently small time interval $[t_0, t_0 + \varepsilon)$, $\varepsilon > 0$. Global existence, for all $t \in \mathbb{R}$ can be shown provided $F$ does not allow "blow-up" ($\|x(t)\| \to \infty$ as $t \to t^*$ for a finite $t^*$).

Definition 2. *[3] Let $K$ be a closed subset of $\mathbb{R}^n$. A trajectory of the differential inclusion (1), $t \to x(t)$, is said to be viable (in $K$) when for all $t$, $x(t) \in K$. The viability kernel of Equation (1) in $K$ is $Viab_K(F)$, the set of initial conditions $x_0 \in K$ such that there exists a solution of $S_F(x_0)$ staying forever in $K$.*

A closed set $K \subset \mathbb{R}^n$ being given, we study the following problem of the existence of trajectories for the differential inclusion (1) remaining in $K$: is $Viab_F(K)$ not empty? That is, does there exist $x_0$ in $K$ and $x(\cdot) \in S_F(x_0)$ such that $\forall t \geq 0, x(t) \in K$? It is well known that the problem has a positive answer for any $x_0 \in K$, and all trajectories, when the boundary of $K$ is the level set of a Lyapunov function associated with the differential inclusion [9]. But finding such Lyapunov functions is generally difficult. The Viability Theorem is a slight relaxation of this approach, to prove that there exists a trajectory staying inside $K$, whereas all trajectories may not stay inside $K$. Let us denote by $C_K(x)$ the Bouligant contingent cone of $K$ at $x$ [3], which, in the case where $K$ is a closed convex subset of $\mathbb{R}^n$ is just the closure of the tangent cone of $K$ at $x$, $\bigcup_{h>0} \left\{ \frac{k-x}{h} \,|k \in K \right\}$.

Theorem 1. *[3] Consider a Marchaud map $F : \mathbb{R}^n \to \wp(\mathbb{R}^n)$ and a closed convex $K \subseteq \mathbb{R}^n$. Suppose that $\forall x \in K$, $F(x) \cap C_K(x) \neq \emptyset$, then $Viab_K(F) = K$, i.e. there always exists a trajectory for the differential inclusion (1) from any point of $K$, staying in $K$.*

The idea behind this theorem is that if there is always a vector field which points inside $K$ in $F(x)$, for all $x \in K$, then there is a way to follow it to stay inside $K$. In this paper, we are going to generalize this approach using a finer characterization of the *exit set* of the differential inclusion $F$. Let $K^S(F) := \{x_0 \in \partial K \mid \forall x \in S_F(x_0) :$ $x$ leaves $K$ immediately$\}$, be the exit set for the differential inclusion $F$. Here, "immediately" means that for every $\varepsilon > 0$ there is $0 < t < \varepsilon$ such that $x(t) \notin K$. We now have the following result, which is a Ważewski property for differential inclusions :

Proposition 1. *[5] Let $K$ be a closed convex subset of $\mathbb{R}^n$ and $F$ a Marchaud map. If the set $K^S(F)$ is closed and not connected[1], then $Viab_F(K) \neq \emptyset$.*

This provides us indeed with a generalization of Theorem 1 in that the former deals with the case where $K^S(F)$ is empty, and hence, is closed and not connected. In the sequel, we focus on deriving conditions on differential inclusions to get closed and disconnected exit sets. We then apply these characterizations to prove the existence of switching modes that make a switched system's trajectory stay within some prescribed region of space. Viability is a generalization of invariance properties, that can be used to verify properties of dynamical systems under uncertainties (Section 6.1) or controlled systems (e.g. Section 8), but few methods are available to compute the viability kernel, and they are not tractable. Our goal is thus only to prove non-emptyness of the viability kernel within some fixed region. As an application of this, for instance, knowing that a parameterized (with uncertain parameters in $U$) dynamical system is viable inside $K$ means that the system is controllable with parameters in $U$. Similarly for arbitrary switching systems : if we prove that the viability kernel is not empty within $K$, then we know that there is a switching strategy to stabilize it, i.e. the system is controllable. For those switched systems which have unique solutions within $K$ (which include a large class of practically meaningful systems), our method will prove that there exists a maximal positive invariant set within $K$, which allows for spotting areas in space where the system is stable (and by way of complement, unstable), if $K$ can be made sufficiently small.

## 2.2 Convex polynomial differential inclusions, in convex compact semi-algebraic sets

### 2.2.1 Convex polynomial differential inclusions

For the rest of the article, we will restrict to the case where $F$ is given as the closed convexification of a finite set of polynomial vector fields $f_1, \ldots, f_q$:

$$F(x) = \overline{co}(f_1, \ldots, f_q) \tag{2}$$

[1] With the convention here that the empty set has no connected component, hence is not connected.

where $co(y_1, \dots, y_q)$ is the convex combination of the $q$ vectors $y_1, \dots, y_q$ in $\mathbb{R}^n$ and $\overline{A}$ is the topological closure of $A$ in $\mathbb{R}^n$. For every such $\lambda = (\lambda_1, \dots, \lambda_q)$ we will write $f_\lambda = \sum_{i=1}^{q} \lambda_i f_i$ so that $F(x)$ can be identified with the set of all such $f_{\lambda(x)}$, where $\lambda$ is a continuous function. Such differential inclusions are very well behaved and we will be allowed to apply the results that we recapped in Section 2.1 :

LEMMA 1. *Set functions $F$ of the form given at Equation (2) are Marchaud maps.*

#### 2.2.2 Convex compact semi-algebraic sets

We will also restrict ourselves further by looking for viable solutions in closed convex sets $K \subset \mathbb{R}^n$, which are defined, for some vector $c = (c_1, \dots, c_m) \in \mathbb{R}^m$, by the $m$ polynomials inequalities:

$$(P) \left\{ \begin{array}{lcl} p_1(x_1, \dots, x_n) & \leq & c_1 \\ & \dots & \\ p_m(x_1, \dots, x_n) & \leq & c_m \end{array} \right.$$

We say that $K$ is a *(polynomial) template*. We call *minimal polynomial templates*, the templates $K$ which border $\partial K$ is equal (and not just included as would be generally the case) to $\bigcup_{i=1}^{m} \{x \mid p_i(x) = c_i, p_j(x) \leq c_j \ \forall j \neq i\}$.

#### 2.2.3 Lie derivatives

Before stating results about the viability kernel of the corresponding differential inclusion, we need to introduce some notions that will be necessary, on polynomial differential equations. $\mathbb{R}[x]$ is the ring of polynomials in $x$.

DEFINITION 3. *(Lie derivative and higher-order Lie derivatives). The Lie derivative of $h \in \mathbb{R}[x]$ along the vector field $f = (f^1, \dots, f^n)$ is defined by $\mathcal{L}_f(h) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f^i = \langle f, \nabla h \rangle$. Higher-order derivatives are defined by $\mathcal{L}_f^{(k+1)}(h) = \mathcal{L}_f(\mathcal{L}_f^{(k)}(h))$ with $\mathcal{L}_f^0(h) = h$.*

For polynomial dynamical systems, only a finite number of Lie derivatives are necessary to generate all higher-order Lie derivatives. Indeed, let $h \in \mathbb{R}[x_1, \dots, x_n]$, we recursively construct an ascending chain of ideals of $\mathbb{R}[x_1, \dots, x_n]$ by appending successive Lie derivatives of $h$ to the list of generators : $\langle h \rangle \subseteq \langle h, \mathcal{L}_f^1(h) \rangle \subseteq \dots \subseteq \langle h, \mathcal{L}_f^1(h), \dots, \mathcal{L}_f^{(N)}(h) \rangle$. Since the ring $\mathbb{R}[x]$ is Noetherian [16], this increasing chain of ideals has necessarily a finite length: the maximal element of the chain is called the differential radical ideal of $h$ and will be noted $\sqrt[\mathcal{L}]{\langle h \rangle}$. Its order is the smallest $N$ such that:

$$\mathcal{L}_f^{(N)}(h) \in \langle h, \mathcal{L}_f^{(1)}(h), \dots, \mathcal{L}_f^{(N-1)}(h) \rangle \tag{3}$$

This $N$ is computationally tractable. If we note $N_i$ the order of the ideal $\sqrt[\mathcal{L}]{\langle p_i \rangle}$, then for face $i$ we should compute the successive Lie derivatives until $N_i$. This can be done by testing if the *Gröbner basis* spanned by the derivatives changes. Indeed, two ideals are equal if they have the same reduced Gröbner basis [1]. If we denote by $\mathcal{G}(\{g_1, \cdots, g_n\})$ the Gröbner basis of $\{g_1, \cdots, g_n\}$, the first $n$ s.t. $\mathcal{G}(\{\mathcal{L}_f^{(0)}(p_i), \cdots, \mathcal{L}_f^{(n)}(p_i)\}) = \mathcal{G}(\{\mathcal{L}_f^{(1)}(p_i), \cdots, \mathcal{L}_f^{(n+1)}(p_i)\})$ is equal to $N_i$.

## 3. VIABILITY OF CONVEX DIFFERENTIAL INCLUSIONS IN TEMPLATES

### 3.1 Viability: a first topological approach

There is first a simple characterization of $K^S(F)$ for differential inclusions of the form we consider in this section, along the lines of [8]:

THEOREM 2. *Consider the differential inclusion (2). Let $K$ be a compact minimal polynomial template defined by the set of inequalities $(P)$ and let $N_i^j$ be the order of the differential ideal $\sqrt[\mathcal{L}]{\langle p_i \rangle}$ along $f_j$. If for each face $K_i$ of template $K$ we have $(H_i)$:*

- *for all $k \in \{1, \dots, N_i^j - 2\}$, for all $\lambda \in \mathbb{R}^q$, for all $(x, \lambda)$,* $\Big( p_i(x) = c_i, \forall j \neq i,\ p_j(x) \leq c_j \ \&\ \sum_{u=1}^{q} \lambda_u = 1,$ $\lambda_1, \dots, \lambda_q \geq 0 \ \&\ \mathcal{L}_{f_\lambda}^{(1)}(p_i)(x) = 0, \dots, \mathcal{L}_{f_\lambda}^{(k)}(p_i)(x) = 0 \Big)$ $\implies \mathcal{L}_{f_\lambda}^{(k+1)}(p_i)(x) \geq 0$
- $\Big\{ (x, \lambda) \mid p_i(x) = c_i, \forall j \neq i\ p_j(x) \leq c_j \ \&\ \sum_{u=1}^{q} \lambda_u = 1,$ $\forall u,\ \lambda_u \geq 0 \ \&\ \mathcal{L}_{f_\lambda}^{(1)}(p_i)(x) = \dots = \mathcal{L}_{f_\lambda}^{(N_i^j - 1)}(p_i)(x) = 0 \Big\}$ *is empty.*

*Then $K^S(F)$ is closed and equal to*

$$\bigcup_{i=1}^{m} \bigcap_{\sum_{u=1}^{q} \lambda_u = 1,\ \lambda_1, \dots, \lambda_q \geq 0} \{x \in K_i \mid \mathcal{L}_{f_\lambda}(p_i)(x) \geq 0\}$$

*If furthermore $K^S(F)$ is disconnected, then $Viab_K(F) \neq \emptyset$.*

Note now that $(H_i)$ can be checked by SoS relaxation [17] in the ring of multivariate polynomials $\mathbb{R}[\lambda_1, \dots, \lambda_q, x_1, \dots, x_n]$, as in [8]. As a matter of fact, $\mathcal{L}_{f_\lambda}^{(v)}(p_i)$ is a polynomial in $\mathbb{R}[\lambda_1, \dots, \lambda_q, x_1, \dots, x_n]$, as easily shown by induction on $v$. But this is both an expensive way to solve our problem and a fairly weak condition for solving problem $(P)$. What the theorem says is that $K^S(F)$ is closed when $K^S(\{f_\lambda\})$ is closed for all $\lambda$. As the example below shows, this is far too strict a condition in general, to be applicable.

EXAMPLE 1 (EXAMPLE 2.7 OF [23]). *Let us consider the switched system defined by :*

$$f_1(x, y) = \begin{pmatrix} -y \\ x - y^3 \end{pmatrix} \quad f_2(x, y) = \begin{pmatrix} y \\ -x - y^3 \end{pmatrix}$$

*We consider the differential inclusion in $\mathbb{R}^2$, $F(x, y) = \overline{co}(f_1, f_2)$ and the template $K = [-0.5, 0.5] \times [-0.5, 0.5]$ given by $p_1 = -x$, $p_2 = x$, $p_3 = -y$, $p_4 = y$ and $c_1 = c_2 = c_3 = c_4 = 0.5$. We have:*

$$\begin{array}{lcllcl} \mathcal{L}_{f_1}(p_1) & = & y & \mathcal{L}_{f_2}(p_1) & = & -y \\ \mathcal{L}_{f_1}(p_2) & = & -y & \mathcal{L}_{f_2}(p_2) & = & y \\ \mathcal{L}_{f_1}(p_3) & = & -x + y^3 & \mathcal{L}_{f_2}(p_3) & = & x + y^3 \\ \mathcal{L}_{f_1}(p_4) & = & x - y^3 & \mathcal{L}_{f_2}(p_4) & = & -x - y^3 \end{array}$$

*For instance, for $p_1(x) = c_1$, $\lambda_1 + \lambda_2 = 1$, we get $\mathcal{L}_{f_\lambda}(p_1) = (\lambda_1 - \lambda_2)y = (1 - 2\lambda_2)y$ which is zero for $y = 0$ or $\lambda_2 = \frac{1}{2}$. In the first case, $\mathcal{L}_{f_\lambda}^{(2)}(p_1) = \frac{1}{2}(1 - 2\lambda_2)^2$ and in the second case $\mathcal{L}_{f_\lambda}^{(2)}(p_1) = (1 - 2\lambda_2)y^3 = 0$. Therefore, the 2nd criterion of $(H_i)$ for $i = 1$ in Theorem 2 is not satisfied. But it can be verified that $V(x, y) = x^2 + y^2$ is a common weak Lyapunov function, so that the system is uniformly stable [23]. We will show later on that more refined topological methods can prove the existence of viable trajectories within $K$.*

### 3.2 Viability in templates with one face

In this section, we further characterize $K^S(F)$ for $F$ of the form given by Equation (1), when $K$ is defined by one face only (for example, when $K$ is an ellipsoid).

LEMMA 2.

$$K^S(F) \quad = \bigcap_{\lambda_1,\ldots,\lambda_q \geq 0,\ \sum_{i=1}^{q} \lambda_i = 1} K^S(\{f_\lambda\})$$

*Furthermore, if the template $K$ is defined by a unique polynomial $p_1$, $K^S(F) = \bigcap_{i=1}^{q} K^S(\{f_i\})$*

In the case of a one-face template $K$, we can refine Theorem 2 to the following result:

THEOREM 3. *Consider the differential inclusion (2). Let $K$ be a convex compact minimal polynomial template defined by the set of inequalities $(P)$ and let $N_1^j$ be the order of the differential ideal $\sqrt[\mathcal{L}]{\langle p_1 \rangle}$ along $f_j$. If for the only face $K_1$ of template $K$, for all $j = 1, \ldots, q$ we have $(H_j)$:*

- *for all $k \in \{1, \ldots, N_1^j - 2\}$,*

$$\begin{aligned}\{x \in K_1 \mid \quad & \mathcal{L}_{f_j}^{(1)}(p_1)(x) = 0, \ldots, \mathcal{L}_{f_j}^{(k)}(p_1)(x) = 0, \\ & \forall i \neq j,\ \mathcal{L}_{f_i}^{(1)}(p_1) \geq 0,\ \mathcal{L}_{f_j}^{(k+1)}(p_1)(x) < 0\}\end{aligned}$$

  *is empty.*

- $$\begin{aligned}\{x \in K_1 \mid \quad & \forall i \neq j,\ \mathcal{L}_{f_i}^{(1)}(p_1) \geq 0\ \& \\ & \mathcal{L}_{f_j}^{(1)}(p_1)(x) = 0, \ldots, \mathcal{L}_{f_j}^{(N_1^j-1)}(p_1)(x) = 0\}\end{aligned}$$

  *is empty*

*Then $K^S(F)$ is closed and equal to*

$$\{x \in K_1 \mid \bigwedge_{j=1}^{q} \mathcal{L}_{f_j}^{(1)}(p_1)(x) \geq 0\}$$

*If furthermore $K^S(F)$ is disconnected, then $Viab_F(K) \neq \emptyset$.*

#### *Algorithm (A).*

We can check the conditions of Theorem 3, in a similar way as what was developed in [8], using Sum of Squares optimization [17] and Stengle's nichtnegativstellensatz, for increasing $k$ from 1 to $N_1^j - 2$, for each vector field $f_j$. We determine polynomials $\alpha_n$ $(n = 0, \ldots, k)$, SoS polynomials $\beta_{S,\mu}$ $(S \subseteq \{1, \ldots, j-1, j+1, \ldots, q\}, \mu \in \{0,1\})$ and an integer $l$, such that

$$\sum_{n=0}^{k} \alpha_n \mathcal{L}_{f_j}^{(n)} + \sum_{\substack{S \subseteq \{1,\ldots,j-1,j+1,\ldots,q\} \\ \mu \in \{0,1\}}} \beta_{S,\mu} G_{S,\mu} + \left(\mathcal{L}_{f_j}^{(k+1)}\right)^{2l} = 0 \tag{4}$$

where $G_{S,\mu} = (-\mathcal{L}_{f_j}^{(k+1)})^{\mu} \prod_{s \in S} \mathcal{L}_{f_s}^{(1)}$ for any $S \subseteq \{1, \ldots, j-1, j+1, \ldots, q\}$, $\mu \in \{0,1\}$ and the convention that $\mathcal{L}_{f_j}^{0}(p_1) = c_1 - p_1$. Practically, this is done by bounding the degrees of the polynomials $\alpha_n$ and $\beta_{S,\mu}$ we are looking for, and taking low values for $l$ (in all our examples, we took $l = 1$), the problem can thus be tested by semidefinite programming. An example of application of Theorem 3 to compute the exit set of a one-face template (ball) is given in Section 6.4.

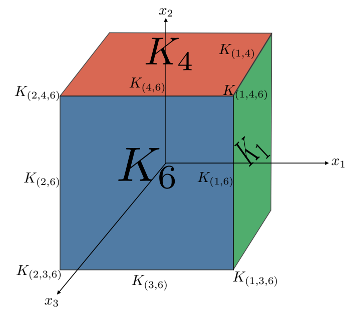


Figure 1: Stratification of a parallellepipedic template.

### 3.3 Viability for general templates

In general, we do not have the same results when the boundary of $K$ is defined by several template polynomials $p_i$, as illustrated in the following example.

EXAMPLE 2. *We carry on with Example 1. We have:* $K^S(\{f_1\}) = [-0.5, 0[ \times \{-0.5\} \cup [0, 0.5[ \times \{0.5\} \cup \{0.5\} \times [-0.5, 0[ \cup \{-0.5\} \times ]0, 0.5]$ *and*
$K^S(\{f_2\}) = [-0.5, 0[ \times \{0.5\} \cup ]0, 0.5] \times \{-0.5\} \cup \{-0.5\} \times [-0.5, 0[ \cup \{0.5\} \times ]0, 0.5]$
*Hence $K^S(\{f_1\}) \cap K^S(\{f_2\})$ is $\{(-0.5, -0.5), (-0.5, 0.5), (0.5, 0.5), (0.5, -0.5)\}$. But at any of the four points above, there is always a vector field within $F(x, y) = \overline{co}(f_1(x, y), f_2(x, y))$ which points strictly inside template $K$. For instance, for point $(0.5, 0.5)$, we have:*

$$\mathcal{L}_{f_\lambda}(p_2) = 0.5(\lambda_2 - \lambda_1) \quad \mathcal{L}_{f_\lambda}(p_4) = 0.5(\lambda_1 - \lambda_2)x - \tfrac{1}{8}$$

*Take for instance $\lambda_1 = \frac{9}{16}$, $\lambda_2 = \frac{7}{16}$, the solutions for $\frac{9}{16} f_1 + \frac{7}{16} f_2$ are entering $K$. The same occurs for the other three points: this is pictured in Figure 2. It follows that $K^S(\{f_\lambda\})$ is empty for $\lambda = (\frac{9}{16}, \frac{7}{16})$, hence $K^S(F)$ is empty (by Lemma 2), hence $K^S(F) \neq K^S(\{f_1\}) \cap K^S(\{f_2\})$.*

Let $K$ now be a convex compact minimal polynomial templates. It is a stratified space. We call $K_i$ the face of the template $K$ given by $\{(x_1, \ldots, x_n) \mid p_i(x_1, \ldots, x_n) = c_i\} \cap K$, and, for all multi-indices $\mathbf{k} \in \{(k_1, \ldots, k_i) \mid 1 \leq k_1 < \ldots < k_i \leq m\}$, $K_{\mathbf{k}}$ the $\mathbf{k}$-face of $K$ given by $K_{\mathbf{k}} = \{x \mid p_{k_1}(x) = c_{k_1}, \ldots, p_{k_i}(x) = c_{k_i}\ \}$. We write $\overset{\circ}{K}_i$ for the interior of the $i$-face, given by $\overset{\circ}{K}_i = \{x \mid p_i(x) = c_i\ \&\ \forall j \neq i, p_j(x) < c_j\}$ and for all multi-indices $\mathbf{k} \in \{(k_1, \ldots, k_i) \mid 1 \leq k_1 < \ldots < k_i \leq m\}$, $\overset{\circ}{K}_{\mathbf{k}}$ is the interior of the $\mathbf{k}$-face of $K$ given by $\overset{\circ}{K}_{\mathbf{k}} = \{x \mid p_{k_1}(x) = c_{k_1}, \ldots, p_{k_i}(x) = c_{k_i}\ \&\ \forall j \notin \mathbf{k}, p_j(x) < c_j\}$

EXAMPLE 3. *Consider template $K$ in $\mathbb{R}^3$ given by $p_1(x) = -x_1$, $p_2(x) = x_1$, $p_3(x) = -x_2$, $p_4(x) = x_2$, $p_5(x) = -x_3$, $p_6(x) = x_3$ and $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 1$. Geometrically, $K$ is the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$, stratified by its 6 faces, 12 edges and 8 extremal points as depicted in Figure 1.*

We have a full characterization of the exit set for $F$ in $K$, which is indeed more involved than Lemma 2. We note $\mathcal{K}_{\mathbf{k}}^S(F) = K^S(F) \cap \overset{\circ}{K}_{\mathbf{k}}$ the intersection of $K^S(F)$ with the interior of face $K_{\mathbf{k}}$.

PROPOSITION 2. *Suppose we have a convex compact minimal polynomial template $K$ defined by the set of inequalities*

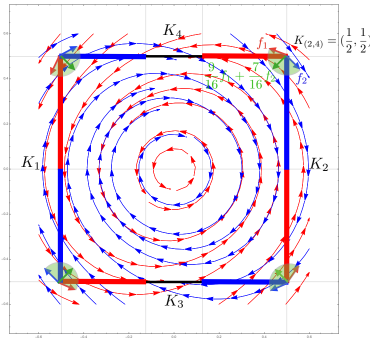


Figure 2: Field $\lambda_1 f_1 + \lambda_2 f_2$ is entrant at $K_{(2,4)}$.

*$(P)$, and consider the differential inclusion for (1). $K^S(F)$ is defined by its intersections with all (iterated) faces of $K$ :*

- *for all $i = 1, \ldots, m$, $\mathcal{K}_i^S(F) = \bigcap_{j=1}^{q} \mathcal{K}_i^S(\{f_j\})$*
- *for all multi-indices $\mathbf{k} = (k_1, \ldots, k_i)$ of cardinality at least 2*
$$\mathcal{K}_{\mathbf{k}}^S(F) = \bigcap_{\lambda_1,\ldots,\lambda_q \geq 0,\ \sum_{i=1}^{q} \lambda_i = 1} \mathcal{K}_{\mathbf{k}}^S(\{f_\lambda\}).$$

Example 4. *We carry on with Example 2. We derive from the calculations of the Lie derivative made in Example 1 that $\mathcal{K}_2^S(\{f_1\}) = \{0.5\} \times [-0.5, 0[$ and $\mathcal{K}_2^S(\{f_2\}) = \{0.5\} \times ]0, 0.5]$. By Proposition 2, we have then:*

$$\mathcal{K}_2^S(F) = \mathcal{K}_2^S(\{f_1\}) \cap \mathcal{K}_2^S(\{f_2\}) = \emptyset \quad \text{and}$$

$$\mathcal{K}_{(2,4)}^S(F) = \bigcap_{\lambda_1,\lambda_2 \geq 0, \lambda_1+\lambda_2=1} \mathcal{K}_{(2,4)}^S(\{\lambda_1 f_1 + \lambda_2 f_2\}) = \emptyset$$

*because of the pair $(\lambda_1, \lambda_2) = \left(\frac{9}{16}, \frac{7}{16}\right)$ found out in Example 2 that makes the vector field $\lambda_1 f_1 + \lambda_2 f_2$ entrant in $K$ in the face $K_{(2,4)} = (0.5, 0.5)$. The three other extremal points of $K$ can be treated in a similar manner, and we conclude that $K^S(F) = \emptyset$. Figure 2 illustrates the entrant field $\lambda_1 f_1 + \lambda_2 f_2$ at $K_{(2,4)}$.*

Unfortunately, the formula for $K^S(F)$ given in Proposition 2 is still highly inconvenient, for two reasons. First, although it is simple to characterize $\mathcal{K}_i^S(F)$ (for all $i = 1, \ldots, m$) in the style of [8], it is hard to characterize $\mathcal{K}_{\mathbf{k}}^S(F)$ as soon as the cardinal of $\mathbf{k}$ is greater than 2. Second, we have to prove that $K^S(F)$ is closed in order to be able to apply Proposition 1 and prove that $Viab_F(K) \neq \emptyset$. The only simple characterization we have at hand is weak: if for all multi-indices $\mathbf{k}$, $\mathcal{K}_{\mathbf{k}}^S(F)$ is closed *in* $\mathbb{R}^n$, then $K^S(F)$ is closed.

We now give a simple criterion for the closedness of $K^S(F)$. The idea is to test closedness of the exit sets on all faces similarly to Proposition 2 and to test whether intersections of faces are entirely exiting or entirely non-exiting (this is called an entrance set or face). In this case, we can easily test whether the exit set is closed by looking only at intersections with each closure of faces $K_i$. Then we will use simple criteria for deciding whether intersections of faces are entirely exiting or entirely non-exiting in Lemmas 3 and 4. Finally, these will have a simple translation in terms of positivstellensatz conditions for some Lie derivative in Theorem 4.

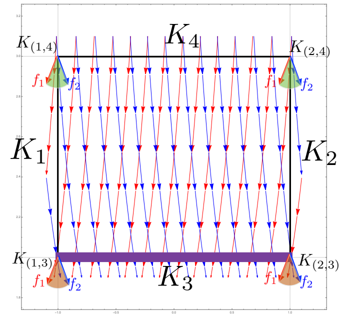


Figure 3: Illustration of Lemmas 3, 4 on Examples 5, 6

For all multi-indices $\mathbf{k} = (k_1, \ldots, k_j)$, let us write $K_{\mathbf{k}}^S(\{f_i\})$ for the exit set of the (not necessarily compact) set $\{x \in \mathbb{R}^n \mid p_{k_1}(x) \leq c_{k_1}, \ldots, p_{k_j}(x) \leq c_{k_j}\}$ under flow $f_i$, intersected with $K_{\mathbf{k}}$.

Proposition 3. *A sufficient condition for $K^S(F)$ to be closed is:*

- *For all $i = 1, \ldots, m$, $\bigcap_{j=1}^{q} K_i^S(\{f_j\})$ is closed in $\mathbb{R}^n$*
- *For all $\mathbf{k}$ multi-index of cardinality at least 2,*
$$\bigcap_{\lambda_1,\ldots,\lambda_q \geq 0,\ \sum_{i=1}^{q} \lambda_i = 1} K_{\mathbf{k}}^S(\{f_\lambda\})$$
*is either empty or the full (iterated) face $K_{\mathbf{k}}$.*

Deciding the second condition of Proposition 3 can be done combinatorially, in simple cases:

Lemma 3. *Consider a point $x \in \overset{\circ}{K}_{\mathbf{k}}$ for some multi-index $\mathbf{k} = (1 \leq k_1 < \ldots < k_i \leq m)$. Suppose that there exists $j$ in $\{1, \ldots, i\}$ such that for all $u$ in $\{1, \ldots, q\}$, $x \in K_{k_j}^S(\{f_u\})$. Then $x$ is in the exit set for $F$, in face $K_{\mathbf{k}}$: $x \in \bigcap_{\lambda_1,\ldots,\lambda_q \geq 0,\ \sum_{i=1}^{q} \lambda_i = 1} K_{\mathbf{k}}^S(\{f_\lambda\})$.*

Example 5. *Consider again Example 1, with the box $K' = [-1, 1] \times [2, 3]$. Simple computations using the corresponding Lie derivatives show that:*

$$\begin{array}{lclclcl} K_3^S(\{f_1\}) & = & [-1, 1] \times \{2\} & K_3^S(\{f_2\}) & = & [-1, 1] \times \{2\} \\ K_j^S(\{f_i\}) & = & \emptyset & \textit{for all} & & i = 1, 2\ j \neq 3 \end{array}.$$

*Figure 3 illustrates this example, where $K^S(F) = [-1, 1] \times \{2\}$ is represented as a thick line. Since $\overset{\circ}{K}_{(1,3)} = \{(-1, 2)\} \in K_3^S(\{f_1\}) \cap K_3^S(\{f_2\})$ and $\overset{\circ}{K}_{(2,3)} = \{(1, 2)\} \in K_3^S(\{f_1\}) \cap K_3^S(\{f_2\})$, Lemma 3 applies and we know that $K_{(1,3)}^S(F) = \{(-1, 2)\}$, $K_{(2,3)}^S(F) = \{(1, 2)\}$. To apply Proposition 3, we need also to sort out whether the 2 other (iterated) faces $K_{(1,4)} = \{(-1, 3)\}$ and $K_{(2,4)} = \{(1, 3)\}$ are entrant or exit faces. This is the objective of the following Lemma.*

We note hereafter $K^T(F)$, the subset of the boundary of template $K$ for which *there exists a solution* to the differential inclusion for (1) which *enters* $K$ immediately. Note that $K^T(F)$ is in the complement of $K^S(F)$. For all multi-indices $\mathbf{k} = (k_1, \ldots, k_i)$, in a similar way as we did for $K^S(F)$, we also note $K_{\mathbf{k}}^T(\{f_i\})$ for such an entrance set for the template $\{x \in \mathbb{R}^n \mid p_{k_1}(x) \leq c_{k_1}, \ldots, p_{k_i}(x) \leq c_{k_i}\}$ under flow $f_i$, intersected with $K_{\mathbf{k}}$. We then have somehow a dual to Lemma 3, with similar proof:

Lemma 4. *Consider a point $x \in \overset{\circ}{K}_{\mathbf{k}}$ for some multi-index $\mathbf{k} = (1 \leq k_1 < \ldots < k_i \leq m)$. Suppose that there exists u in $\{1, \ldots, q\}$ such that for all $j$ in $\{1, \ldots, i\}$, $x \in K_{k_j}^T(\{f_u\})$. Then $x \in K^T(F)$: $x \in \bigcap_{\lambda_1,\ldots,\lambda_q \geq 0,\ \sum_{i=1}^q \lambda_i = 1} K_{\mathbf{k}}^T(\{f_\lambda\})$.*

Example 6. *We carry on with Example 5. Let us look at face $K_{(1,4)} = \{(-1, 3)\}$. We see that $\mathcal{L}_{f_2}(p_1)(-1, 3) = -3 < 0$ and $\mathcal{L}_{f_2}(p_4)(-1, 3) = -26 < 0$, hence $(-1, 3) \in K_{(1,4)}^T(\{f_1, f_2\})$. By Lemma 4, $K_{(1,4)} = (-1, 3)$ is entrant and $K_{(1,4)}^S(F) = \emptyset$. Similarly, we would find $K_{(2,4)}^S(F) = \emptyset$ so now Proposition 3 applies and $K^S(F) = [-1, 1] \times \{2\}$ is closed. This situation is illustrated on Figure 3, where at the corners, always exiting flows (at $K_{(1,3)}$ and $K_{(2,3)}$) are represented in orange, whereas the flows that can possibly enter the template (at $K_{(1,4)}$ and $K_{(2,4)}$) are in green.*

Proposition 3, Lemmas 3, 4, translate into polynomial decision problems, as the theorem below expresses. These polynomial problems derive from similar conditions given for ordinary flows in [8] as we explain below.

Theorem 4. *Consider the differential inclusion for (2). Let $K$ be a convex compact minimal polynomial template defined by the set of inequalities $(P)$ and let $N_i^j$ be the order of the differential ideal $\sqrt[\mathcal{L}]{\langle p_i \rangle}$ along $f_j$. Suppose we have:*

- *For all faces $i$, all $j = 1, \ldots, q$, and all $l = 1, \ldots, N_i^j - 2$, we have $(H_{i,j,l})$:*

$$\big(p_i(x) = c_i \ \&\ \forall k \neq i,\ p_k(x) \leq c_k \ \&\ \mathcal{L}_{f_j}(p_i) = 0, \ldots, \mathcal{L}_{f_j}^{(l)}(p_i) = 0 \ \&\ \forall v \neq j,\ \mathcal{L}_{f_v}(p_i) \geq 0\big) \Rightarrow \mathcal{L}_{f_j}^{(l+1)}(p_i)(x) \geq 0$$

  *and $(H_{i,j,N_i^j-1})$: the following set is empty*

$$\{x \in K_i \mid \quad \forall k \neq i,\ \mathcal{L}_{f_k}^{(1)}(p_i) \geq 0 \ \& \ \mathcal{L}_{f_j}^{(1)}(p_i)(x) = 0, \ldots, \mathcal{L}_{f_j}^{(N_i^j-1)}(p_i)(x) = 0\}$$

- *For all multi-indices $\mathbf{k}$ in $\{1, \ldots, m\}$, of cardinality greater or equal than 2,*
  - *there exists $l = 1, \ldots, i$ s.t. we have $(H_{l,\mathbf{k}}^+)$:*

$$\begin{aligned}&(p_{k_1}(x) = c_{k_1} \ \& \ \ldots \ \& \ p_{k_i}(x) = c_{k_i} \& \\ &\forall k \neq \{k_1, \ldots, k_i\},\ p_k(x) \leq c_k) \\ &\Longrightarrow \big(\mathcal{L}_{f_1}(p_{k_l})(x) > 0 \ \& \ \ldots \ \& \ \mathcal{L}_{f_q}(p_{k_l}(x) > 0\big)\end{aligned}$$

  - *Or there exists $j = 1, \ldots, q$ such that $(H_{j,\mathbf{k}}^-)$:*

$$\begin{aligned}&(p_{k_1}(x) = c_{k_1} \ \& \ \ldots \ \& \ p_{k_i}(x) = c_{k_i} \& \\ &\forall k \neq \{k_1, \ldots, k_i\},\ p_k(x) \leq c_k) \\ &\Longrightarrow \big(\mathcal{L}_{f_j}(p_{k_1})(x) < 0 \ \& \ \ldots \ \& \ \mathcal{L}_{f_j}(p_{k_i})(x) < 0\big)\end{aligned}$$

*Then $K^S(F)$ is closed and is $\bigcup_{i=1}^m \{x \in K_i \mid \bigwedge_{j=1}^q \mathcal{L}_{f_j}^{(1)}(p_i)(x) \geq 0\}$. Also, if $K^S(F)$ is disconnected, then $Viab_K(F) \neq \emptyset$.*

Conditions $(H_{i,j,l})$ ensure that at a point of tangency of flow $f_j$, on face $K_i$, of order $l$, which is also in the exit set of the other flows $f_v$, a certain $(l+1)$th Lie derivative is non-negative, meaning that flow $f_j$ at that point is still exiting – this ensures closedness of the exit set on this face. Conditions $(H_{l,k}^+)$ (resp. $(H_{l,k}^-)$ ensure that the exit sets on $K_{\mathbf{k}}$ are the full face (resp. are empty).

***Algorithm (B).***
In the same way as with the algorithm of Theorem 3, we can check the conditions of Theorem 4 by using Sum of Squares optimization [17] and Stengle's nichtnegativstellensatz, for increasing $k$ from 1 to $N_i^j - 2$ and $j$ from 1 to $q$. In practice, as in [8], we generally use only a sufficient condition $(H_{i,j})$, which implies all $(H_{i,j,l})$:

$$\begin{aligned}&\big(p_i(x) = c_i \ \&\ \forall k \neq i,\ p_k(x) \leq c_k \ \&\ \mathcal{L}_{f_j}(p_i) = 0 \ \& \\ &\forall l \neq j,\ \mathcal{L}_{f_l}(p_i) \geq 0\big) \Longrightarrow \mathcal{L}_{f_j}^{(2)}(p_i)(x) > 0\end{aligned}$$

and which can be checked using the much less computationally demanding Putinar positivstellensatz. When conditions $(H_{l,k}^+)$ or $(H_{l,k}^-)$ are not satisfied, we use the characterization of Proposition 3 translated as in Theorem 2, as a polynomial decision problem on variables $(x, \lambda)$. This will be exemplified in Example 8.

We are now going to explain how Theorem 4 can be applied for proving the existence of viable solutions for time-dependent switched systems (Section 4) and state-dependent switched systems (Section 5).

# 4. VIABILITY AND INVARIANTS OF TIME-DEPENDENT SWITCHED SYSTEMS

Let us recall the notions related to *time-dependent switched system* (see, e.g., [18]). Suppose that we are given a family $f_i$, $i \in Q = \{1, \ldots, q\}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. The set $Q$ is called the set of *modes*. We still assume here that the functions are polynomials (hence locally Lipschitz). Let $G$ defined on every point $x$ of $\mathbb{R}^n$ by $G(x) = \{f_1(x), \ldots, f_q(x)\}$. It has closed, non-empty values, and is locally-Lipschitz, hence [3], the corresponding differential inclusion

$$\dot{x} \in G(x) \tag{5}$$

has solutions over finite time intervals. A solution of such a differential inclusion is any absolutely continuous functions satisfying $\dot{x}(t) \in G(x(t))$ almost everywhere. Such functions define time-dependent trajectories of the switched systems with the $q$ modes $f_1, \ldots, f_q$.

A classical way to study the switched system (5) is to consider instead the differential inclusion equation $\dot{x} \in F(x)$ where $F$ is defined by (2). Indeed, the Filippov-Ważewski theorem, which is basically a generalisation of the bang-bang control in ordinary linear control, states that all solutions of the convexified equation (2) can be approximated by solutions of Equation (5) with the same initial value, at least over a compact time interval, and under some simple hypotheses.

But we will actually need a little more than this classical theorem if we want to use the results of the previous section for switched systems with time-dependent switching. There are ways to extend it to infinite time horizon, still keeping some control over the switched trajectories, with respect to the trajectories of the corresponding differential inclusion, at the expense of possibly having to slightly perturbate the initial condition [13]. We restrict this version of Filippov-Ważewski "in infinite horizon" to our case, where we study differential inclusions $F = \overline{co}(f_1, \ldots, f_q)$ over $\mathbb{R}^n$, which are autonomous (they do not depend on time). In what follows, let $B(x, R)$ be the Euclidean ball in $\mathbb{R}^n$ of center $x$ and radius $R$, and $d_H$ the Hausdorff distance.

Theorem 5. *[13] Let $0 < T \leq \infty$. Suppose the set-valued map $G : \mathbb{R}^n \to \wp(\mathbb{R}^n)$ is measurable with respect to the Borel subsets of $\mathbb{R}^n$. Suppose also that for all $R > 0$ there exists $k_R \in \mathbb{R}$ such that for any $\xi, \eta \in B(0, R)$, $d_H(G(\xi), G(\eta)) \leq k_R|\xi - \eta|$ and that there exists $\alpha_R \in \mathbb{R}$ such that for each $\xi \in B(0, R)$, $\sup\{|\zeta| : \zeta \in G(\xi)\} \leq \alpha_R$ . Fix $\xi \in X$ and let $z \in [0, T) \to X$ be a solution of $\dot{x} \in \overline{co}(G(x))$, $x(0) = \xi$. Let $r = [0, T) \to \mathbb{R}$ be a continuous function satisfying $r(t) > 0$ for all $t \in [0, T]$. Then there exists $\eta^0 \in B(\xi, r(0))$ and a solution $x = [0, T) \to X$ of $\dot{x} \in G(x)$, $x(0) = \eta^0$ which satisfies $|z(t) - x(t)| \leq r(t)$ $\forall t \in [0, T)$.*

Lemma 5. *The switched system (5) satisfies the hypotheses of Theorem 5.*

We are now in a position to use Theorem 5 for the differential inclusion $\overline{co}(f_1, \ldots, f_q)$. We can prove the following adaptation of Theorem 4 to time-dependent switched systems:

Theorem 6. *Consider the time-dependent switched system of Equation (5). Let $K$ be a compact minimal polynomial template defined by the set of inequalities $(P)$ and let $N_i^j$ be the order of the differential ideal $\sqrt[\mathcal{L}]{\langle p_i \rangle}$ along $f_j$. Under the same conditions as those of Theorem 4 $K^S(F)$ is closed and equal to $\bigcup_{i=1}^{m} \{x \in K_i \mid \bigwedge_{j=1}^{q} \mathcal{L}_{f_j}^{(1)}(p_i)(x) \geq 0\}$. If furthermore $K^S(F)$ is disconnected, then for any open set $\hat{K}$ strictly containing $K$, $Viab_G(\hat{K}) \neq \emptyset$, i.e.: there exists a viable solution of (5) within $\hat{K}$.*

Example 7. *Consider again the switched system of Example 1. >From Example 5, we know that for template $K' = [-1, 1] \times [2, 3]$, $K^S(F) = [-1, 1] \times \{2\}$. But this exit set is connected, and Theorem 6 does not apply. It is actually clear that there is no switching that can stabilize $F$ within $K'$: any infinite trajectory of the convexified flow of $f_1$ and $f_2$ either does not intersect with $K'$, or traverses $K'$ (i.e., enters into $K'$ then exits from it), as shown in Figure 3.*

*We consider now the same switched system, but with the box $K = [-0.5, 0.5] \times [-0.5, 0.5]$. We know from Example 4 that $K^S(F) = \emptyset$ and Theorem 6 applies. We can then conclude that there exists a time-dependent switching which stabilizes $F$ within e.g. any square $K_\epsilon = ]-0.5 - \epsilon, 0.5 + \epsilon[ \times ]-0.5 - \epsilon, 0.5 + \epsilon[$ $(\epsilon > 0)$, by choosing appropriately $f_1$ or $f_2$ for some amount of time. Actually, there is a trivial switching: both systems $f_1$ and $f_2$ stabilize within $K$ (this is clear from Figure 2). Note however that the topological criterion given in [8] is too weak to conclude, as both exit sets $K^S(\{f_1\})$ and $K^S(\{f_2\})$ are non-closed.*

# 5. VIABILITY AND INVARIANTS OF STATE-DEPENDENT SWITCHED SYSTEMS

Switching events often depend not only on time, but also on the current state of the system (see [18]). Suppose we are given a partition of $\mathbb{R}^n$ as a finite or infinite number of *operating regions* by means of a family of *switching surfaces*, or *guards*. A *state-dependent switched system* is defined by these operating regions, and in the interior of each of these regions a continuous dynamical system. Whenever the system trajectory hits a switching surface, the continuous state changes its *mode*. For simplicity, we suppose hereafter that there are only 2 modes and 1 switching surface. The generalization is easy.

Consider a state-dependent switched system, described by a $\mathcal{C}^1$ switching surface $\mathcal{S}$, given by equation $s(x) = 0$ separating $\mathbb{R}^n$ into two open components $\mathcal{S}_+ = \{x \in \mathbb{R}^n \mid s(x) > 0\}$ and $\mathcal{S}_- = \{x \in \mathbb{R}^n \mid s(x) < 0\}$, and two subsystems $\dot{x} = f_i(x)$, $i = +, -$, one on each side of each element of $\mathcal{S}$:

$$\dot{x} = \begin{cases} f_+(x) & \text{if } s(x) > 0 \\ f_-(x) & \text{if } s(x) < 0 \end{cases} \tag{6}$$

We rely on Filippov's definition of a solution to such systems :

Definition 4. *[18] Given a state-dependent switched system $\mathcal{H}$ defined by Equation 6, a function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is a* solution *of $\mathcal{H}$ if it is absolutely continuous and satisfies the differential inclusion $\dot{x}(t) \in F(x(t))$ for almost all $t \in \mathbb{R}^+$, where $F$ is a multi-valued function defined as follows:*

$$F(x) \quad = \quad \begin{cases} \{f_-(x)\} & \text{if } x \in \mathcal{S}_- \\ \{f_+(x)\} & \text{if } x \in \mathcal{S}_+ \\ \overline{co}\{f_+(x), f_-(x)\} & \text{if } x \in \mathcal{S} \end{cases}$$

Similarly to Lemma 1, function $F$ from Definition 4 is a Lipschitzean Marchaud map, hence admits solutions on finite time intervals (see [7], Chapter 2). We apply again Theorem 5 and get a theorem similar to Theorem 6 :

Theorem 7. *Consider the state-dependent switched system defined by Equation (6). Let $K$ be a compact minimal polynomial template defined by the set of inequalities $(P)$. Suppose the switching surface $\mathcal{S}$ intersects $K$ only at intersections of faces defining $K$, i.e. is entirely within* **k***-faces (with $|\, \mathbf{k} \,| \geq 2$) of $K$. Suppose, up to a reordering of faces, that $\{p_i \mid i = 1, \ldots, l\}$ (resp. $\{p_j \mid j = l+1, \ldots, m\}$) are the polynomials defining the faces of $K$ whose interior are in $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) and let $N_i^j$ be the order of the differential ideal $\sqrt[\mathcal{L}]{\langle p_i \rangle}$ along $f_j$. Under the same conditions as those of Theorem 4, we have: $K^S(F)$ is closed and equal to $\bigcup_{i=1}^{l} \{x \in K_i \mid \mathcal{L}_{f_+}^{(1)}(p_i)(x) \geq 0\} \cup \bigcup_{j=l+1}^{m} \{x \in K_j \mid \mathcal{L}_{f_-}^{(1)}(p_j)(x) \geq 0\}$. If furthermore $K^S(F)$ is disconnected, then $Viab_F(\hat{K}) \neq \emptyset$ ($\hat{K}$ is any open set strictly containing $K$), i.e.: there exists a state-dependent switching signal for which there is a viable solution of (6) within $\hat{K}$.*

# 6. EXPERIMENTS

## 6.1 An uncertain differential system

We consider a perturbation of the system discussed in [8] :

$$f_\epsilon \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^3 + y - x/10 + \varepsilon \\ -x - y/10 + \varepsilon \\ 5z + \varepsilon \end{pmatrix}$$

and we take as differential inclusion $F(x,y,z) = \{f_\epsilon \mid -0.05 \leq \varepsilon \leq 0.05\}$. We consider the template given by the unique face $p_1 = x_1^2 + (x_2-1)^2 + (x_3+1)^2$ and $c_1 = 1/25$. We use Theorem 2 and its implementation as Algorithm (A). With our Matlab implementation, the exit set for the differential inclusion is proved closed in 348 seconds. It has two connected components hence there is a viable trajectory within the template considered.

## 6.2 Boost DC-DC Converter

EXAMPLE 8. *The boost DC-DC converter is an example from power electronic, where the state of the system is* $x(t) = [i_l(t)\ v_c(t)]^T$ *with* $i_l$ *the current intensity in an inductor, and* $v_c(t)$ *the voltage of a capacitor. The aim of the control is to maintain the system inside a given zone* $K$ *(while the output voltage stabilizes around a desired value). The dynamics associated with mode* $u$ *is given by* $\dot{x}(t) = f_u(x) = A_u x(t) + b$ *(*$u = 1,2$*) with* $b = (\frac{v_s}{x_l}\ 0)^T$*. We use in the experiments the numerical values of [11] for* $A_1$ *and* $A_2$*. and study the system in the rectangle* $K = [1.55, 2.15] \times [1.0, 1.4]$*, which corresponds to the template* $p_1 = x$*,* $p_2 = -x$*,* $p_3 = y$*,* $p_4 = -y$ *with* $c_1 = 2.15, c_2 = -1.55, c_3 = 1.4, c_4 = -1$.

### *Time-dependent switching.*

The Lie derivatives for each dynamic and each face, instantiated for the chosen parameters and template $K$, are given below

| | |
|---|---|
| $\mathcal{L}_{f_1}(p_1) = -0.017x + .3 > 0$ | $\mathcal{L}_{f_2}(p_1) = -0.018x - 0.33y + .3 < 0$ |
| $\mathcal{L}_{f_1}(p_2) = -L_{f_1}(p_1) < 0$ | $\mathcal{L}_{f_2}(p_2) = -\mathcal{L}_{f_2}(p_1) > 0$ |
| $\mathcal{L}_{f_1}(p_3) = -0.014y < 0$ | $\mathcal{L}_{f_2}(p_3) = 0.014x - 0.014y > 0$ |
| $\mathcal{L}_{f_1}(p_4) = -L_{f_1}(p_3) > 0$ | $\mathcal{L}_{f_2}(p_4) = -\mathcal{L}_{f_2}(p_3) < 0$ |

The conditions of Theorem 6 are satisfied, using the simpler version of Algorithm B, for all faces $K_i$: actually all first-order Lie derivatives are strictly positive on the faces. This is checked in 12.22 seconds using our Matlab implementation. The conditions of Theorem 6 are not satisfied on the two extremal points $K_{(1,3)} = (1.55, 1.0)$ and $K_{(2,4)} = (2.15, 1.4)$ where we use the more complex characterization of Proposition 3 as explained in Algorithm (B), showing the these two points are not part of the exit set (see also Figure 4). Therefore $K^S(F) = \emptyset$ and there exists a viable solution in any open set containing $K$.

### *State-dependent switching.*

We now consider the same system with a switching surface $S$ given by the affine function going through the corners (1.55, 1.0) and (2.15, 1.4) i.e. $\{s(x) = y - ax - b = 0\}$ where $a = 2/3$ and $b = -1/30$, and $f_1$ is applied in $\{s(x) > 0\}$ and $f_2$ in $\{s(x) < 0\}$ (see Figure 4).

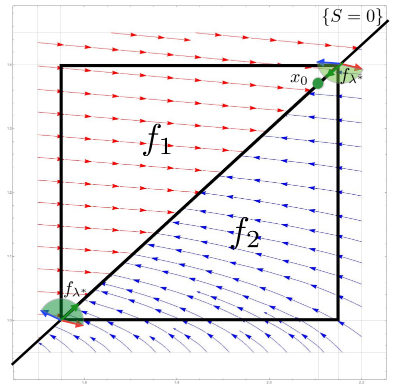


Figure 4: Flows for Example 8 with switching surface S

Here again, we can show that $K^S(F)$ is empty using Theorem 7. It is clear indeed that all the points of $\mathcal{S}_- \cap \partial K$ and $\mathcal{S}_+ \cap \partial K$ are entrant and satisfy the conditions of Theorem 7 (i.e., those of Theorem 4). For the corners located on $\mathcal{S}$, this is less trivial (similarly to what happens in the time-dependent switching case) since $f_1$ and $f_2$ are both exiting. But there are values of $\lambda$ for which flow $f_\lambda$ is entrant. [2]

## 6.3 Defocused switched systems

EXAMPLE 9. *Let us consider the defocused switched system defined by*

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\rho_A & -1/E \\ E & -\rho_A \end{pmatrix} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \quad (7)$$

$$f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\rho_B & -1 \\ 1 & -\rho_B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (8)$$

*The invariant sets of such systems are studied in [22]. We consider here the case* $(x_c, y_c) = (cos(\phi), sin(\phi))$*,* $\rho_A = 0.5, \rho_B = 0.4, E = 0.5, \phi = 0$*. The flows for the two systems are represented in Figure 6.*

We choose box templates centered on the $x$ axis, defined by $p_1 = x$, $p_2 = -x$, $p_3 = y$, $p_4 = -y$ and $c_1 = c_0 + \delta_1$, $c_2 = -c_0 + \delta_1$, $c_3 = c_4 = \delta_2$ (see Figure 6). Calculating the first-order Lie derivative for each dynamic and face, we see that the boxes have an empty exit set $K_S(F)$, and it follows from Theorem 6 that the state of the system can be maintained inside any box using an appropriate switching law. For example the system can be controlled inside the box defined by $(c_0 = 0.4, \delta_1 = 0.1, \delta_2 = 0.15)$ (see Figure 5) and $(c_0 = 0.55, \delta_1 = 0.05, \delta_2 = 0.15)$, hence fairly accurately.

## 6.4 Disconnected exit sets

EXAMPLE 10. *We consider the switched system defined by*

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + y^2 \\ y \end{pmatrix} \text{ and } f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 - y^2 \\ y \end{pmatrix}$$

[2]Actually, the surface $\mathcal{S}$ can be seen as a *sliding* surface, i.e. there exists a solution which stays indefinitely on it: the sliding strategy makes all the trajectories starting from $K$ converge to an equilibrium point $(2.10648, 1.37098)$, represented as a green bullet point on Figure 4.

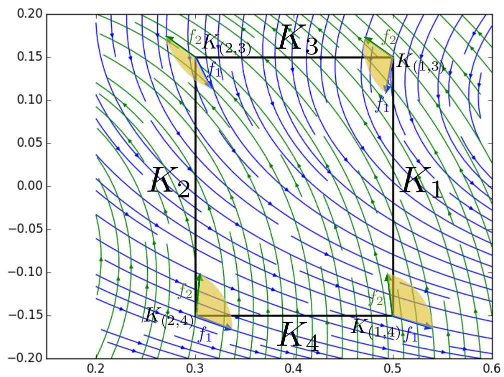


Figure 5: $K^S = \emptyset$ for template $(c_0, \delta_1, \delta_2) = (0.4, 0.1, 0.15)$.

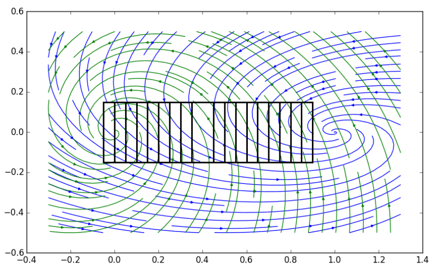

Figure 6: Box templates between the equilibrium points of the defocused switched system.

### *Box template.*

We consider the template $p_1 = -x$, $p_2 = x$, $p_3 = -y$, $p_4 = y$ and $c_1 = c_2 = c_3 = c_4 = 1$. Calculating the Lie derivative of the template, we can deduce closedness of the exit set by Theorem 6, and, by Algorithm (B), in 22 seconds in Matlab, $K^S(F) = [-1, 1] \times \{-1\} \cup [-1, 1] \times \{1\}$, which is closed and disconnected (left of Figure 7).

### *Ball template.*

Let us now take a unit ball template: $p = x^2 + y^2$ and $c = 1$. Calculating the Lie derivatives on this template we can deduce, using Theorem 6 and Algorithm (B), in 8.6 seconds using our Matlab implementation that $K^S(F) = \{(x, y) \in [-0.445042, 0.445\ 042] \times [-1, 1] \mid x^2 + y^2 = 1\}$, which is closed and disconnected (right of Figure 7). Hence the ball as well as the box contain a viable trajectory for some time-dependent switching strategy.

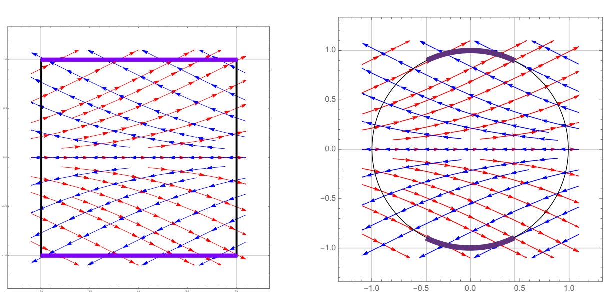

Figure 7: Flows $f_1$ and $f_2$, with $K^S(F)$ in thick line.

EXAMPLE 11. *Consider now the following generalization of the previous system, to dimension 3 :*

$$f_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + y^2 + z^2 \\ y \\ z \end{pmatrix}, f_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 - y^2 - z^2 \\ y \\ z \end{pmatrix},$$

$$f_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 1 + x^2 + z^2 \\ z \end{pmatrix}, f_4 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -1 - x^2 - z^2 \\ z \end{pmatrix}$$

### *Ball template.*

We use a ball template defined by $p = x^2 + y^2 + z^2$ and $c = 1$. By application of Theorem 6, with just the first two Lie derivatives as in Algorithm (B), we find in 68.5 seconds using our Matlab implementation that the exit set is closed. It is made of two components (see Figure 8, the components are in red) $K^S(F) = \{(x, y, z) \in [-0.445042, 0.445042] \times [-0.445042, 0.445042] \times [-1, 1] \mid x^2 + y^2 + z^2 = 1\}$ : there exists a time-dependent switch stabilizing this system of four non-linear ODEs in the unit ball of dimension 3.

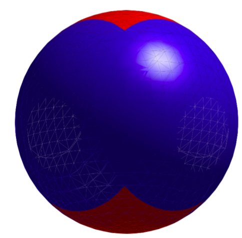

Figure 8: The two components of the exit set, Example 11

## 7. CONCLUSION AND FUTURE WORK

We have explained in this paper how topological methods can be used in order to show the presence of invariant sets inside given templates of the phase space of switched systems. Computable criteria based on SoS methods have been given, and successfully experimented on various examples of differential inclusions and time-dependent and state-dependent switched systems of the literature. We think that our approach sheds new light on the important problem of locating invariants of switched systems. It is now natural to consider *parametric* templates of a given form, and determine the values of the parameters which satisfy our criteria: this will allows us to *synthesize* templates containing invariants. As future work, we plan to apply more refined topological methods based on the Conley index, in order to determine the dynamical nature of located invariants (stable or unstable equilibrium point, limit cycle, chaos,...).


## 8. ACKNOWLEDGMENTS

The authors were partially supported by Digiteo Project SIMS 2013-0544D, by ANR projects CAFEIN, ANR-12-INSE-0007 and MALTHY, ANR-13-INSE-0003, by iCODE and by the academic chair “Systems Engineering” of Ecole polytechnique-ENSTA-Télécom-Thalès-Dassault Aviation-DCNS-DGA-FX-Fondation ParisTech-FDO ENSTA.


## 9. REFERENCES

[1] W. W. Adams and P. Loustaunau. *An introduction to Grobner bases*. Graduate studies in mathematics. American mathematical society, 1994.

[2] E. Asarin, O. Bournez, D. Thao, O. Maler, and A. Pnueli. Effective synthesis of switching controllers for linear systems. *Proc. of the IEEE*, 88(7):1011–1025, July 2000.

[3] J.-P. Aubin and A. Cellina. *Differential Inclusions, Set-Valued Maps And Viability Theory*. Number 264 in Grundl. der Math. Wiss. Springer, 1984.

[4] M. Boccadoro, Y. Wardi, M. Egerstedt, and E. Verriest. Optimal control of switching surfaces in hybrid dynamical systems. *Discrete Event Dynamical Systems*, 15:433–448, 2005.

[5] P. Cardaliaguet. Conditions suffisantes de non-vacuité du noyau de viabilité. *C. R. Acad. Sci. Paris Ser. I*, 314:797–800, 1992.

[6] S. Coogan and M. Arcak. Guard synthesis of hybrid systems using sum of squares programming. In *Conference on Decision and Control*, 2012.

[7] A. F. Filippov. *Differential equations with discontinuous righthand sides*. Mathematics and its applications. Kluwer, 1988.

[8] L. Fribourg, É. Goubault, S. Mohamed, M. Mrozek, and S. Putot. A topological method for finding invariants of continuous systems. In *Workshop on Reachability Problems*, volume 9328 of *LNCS*. Springer, 2015.

[9] G. Gabor and M. Quincampoix. On existence of solutions to differential equations or inclusions remaining in a prescribed closed subset of a finite-dimensional space. *Journal of Differential Equations*, 185:483–512, 2002.

[10] A. Girard and G. Pappas. Approximated bisimulation relations for constrained linear systems. *Automatica*, 43:1307–1317, 2007.

[11] A. Girard, G. Pola, and P. Tabuada. Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions on Automatic Control*, 55:116–126, 2010.

[12] E. Haghverdi, P. Tabuada, and G. Pappas. Bisimulation relations for dynamical, control, and hybrid systems. *Theor. Comput. Sci.*, 342, 2005.

[13] B. Ingalls, E. D. Sontag, and Y. Wang. An infinite-time relaxation theorem for differential inclusions. *Proceedings of the AMS*, 131(2), 2002.

[14] S. Jha, S. Gulwani, S. Seshia, and A. Tiwari. Synthesizing switching logic for safety and dwell-time requirements. In *ACM/IEEE Int. Conf. on Cyber-Physical Systems*, 2010.

[15] A. Julius and A. van der Schaft. The maximal controlled invariant set of switched linear systems. In *Conference on Decision and Control*. IEEE, 2002.

[16] S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2002.

[17] J.-B. Lasserre. *Moments, positive polynomials and their applications*, volume 1. World Scientific, 2009.

[18] D. Liberzon. *Switching in Systems and Control*. Systems and Control: Foundations and Applications. Birkhauser, 2003.

[19] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35:349–370, 1999.

[20] K. Mischaikow and M. Mrozek. Conley index theory. *Handbook of Dynamical Systems II*, 2002.

[21] R. Moeckel. Sturm's algorithm and isolating blocks. *Journal of Symbolic Computation*, 40:1242–1255, 2005.

[22] P. Nilsson, U. Boscain, M. Sigalotti, and J. Newling. Invariant sets of defocused switched systems. In *Conference of Decision and Control*, 2013.

[23] Z. Sun and S. S. Ge. *Stability Theory of Switched Dynamical Systems*. Springer, 2011.

[24] C. Tomlin, J. Lygeros, and S. Sastry. A game theoretic approach to controller design for hybrid systems. In *Proceedings of the IEEE*, 2000.