Tree Automata and Applications Exercise session 4 Solutions

Luc Lapointe luc.lapointe@ens-paris-saclay.fr home.lmf.cnrs.fr/LucLapointe/

Exercise 1 - WS2S on finite trees

We consider trees with maximum arity 2. Give WS2S formulae which express the following:

1. $x \subseteq y$, with \subseteq the prefix relation on positions.

Detailed solution

$$x \subseteq y \coloneqq \forall X (y \in X \land (\forall z (z1 \in X \lor z2 \in X) \Rightarrow z \in X)) \Rightarrow x \in X$$

Double inclusion proof.

Let *y* a tree. Prove by induction on *x* than if *x* is a prefix of *y*, then $x \subseteq y$.

Let x, y such that x is not a prefix of y. Then the set of y and all of its prefixes is an example that $x \nsubseteq y$.

2. X is closed under predecessors.

Solution

$$\mathrm{closed}(X)\coloneqq \forall z\forall y(x\in X\wedge y\subseteq x)\Rightarrow y\in X$$

3. The letter *a* occurs twice on the same path.

Solution

$$\exists x \exists y (\neg (x = y) \land x \subseteq y \land P_a(x) \land P_a(y))$$

4. The letter *a* occurs twice not on the same path.

Solution

$$\exists x \exists y (\neg (x \subseteq y) \land \neg (y \subseteq x) \land P_a(x) \land P_a(y))$$

5. There exists a subtree with only a letters.

Solution $\exists x \forall y (x \subseteq y \Rightarrow P_a(y))$

Exercise 2 - The limit of WSkS

Prove that the predicate x = 1y is not definable in WSkS.

Solution

Assume x = 1y is definable in WSkS. Then there exist a tree automaton recognizing it.

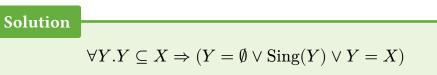
Consider the family of positions 12^n . In the tree describing $x = 12^n$ and $y = 2^n$, x goes left from the root then only right, and y goes only right from the root. This tree is in the language of the WSkS formula.

Using a refined version of the pumping lemma, as for example in the Exercise session 1 correction, we can increase the size of one of those two branches to reach a contradiction.

Exercise 3 - The power of WSkS

Produce formulae of WSkS for the following predicates:

1. The set X has exactly two elements.



where $\operatorname{Sing}(Y)$ denotes singletons:

$$\operatorname{Sing}(X) \stackrel{\text{def}}{=} X \neq \emptyset \land \forall Y (Y \subseteq X \Rightarrow (Y = X \lor Y = \emptyset))$$

2. The set X contains at least one string beginning with a 1.

Solution

```
\exists x. x \in X \land 1 \le x
```

3. Given a formulae of WSkS φ with one free first-order variable, produce a formula of WSkS expressing that there is an infinity of words on $\{1, ..., k\}^*$ satisfying φ .

Solution

Define $X \vDash \varphi$ by $\forall x, x \in X \Rightarrow \varphi(x)$. Now the wanted formula:

 $\forall X, X \vDash \varphi \Rightarrow \exists Y, X \subset Y \land Y \vDash \varphi$

Homework - From formulae to automata

Give tree automata recognizing the languages on trees of maximum arity 2 defined by the following formulae:

- $\begin{array}{l} 1. \ (x \in S \land (x1 = y \Rightarrow y \in S)) \land \left(z \in S \Rightarrow P_f(z)\right) \\ 2. \ \exists S \big[(x \in S \land (x1 = y \Rightarrow y \in S)) \land \left(z \in S \Rightarrow P_f(z)\right) \big] \end{array}$