# Tree Automata and Applications

Exercise session 4 Solutions

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## Exercise 1 - WS2S on finite trees

We consider trees with maximum arity 2. Give WS2S formulae which express the following:

1. $x \subseteq y$, with $\subseteq$ the prefix relation on positions.

**Detailed solution**

$$x \subseteq y := \forall X(y \in X \land (\forall z(z1 \in X \lor z2 \in X) \Rightarrow z \in X)) \Rightarrow x \in X$$

Double inclusion proof.

Let $y$ a tree. Prove by induction on $x$ than if $x$ is a prefix of $y$, then $x \subseteq y$.

Let $x, y$ such that $x$ is not a prefix of $y$. Then the set of $y$ and all of its prefixes is an example that $x \nsubseteq y$.

2. $X$ is closed under predecessors.

**Solution**

$$\text{closed}(X) := \forall z \forall y(x \in X \land y \subseteq x) \Rightarrow y \in X$$

3. The letter $a$ occurs twice on the same path.

**Solution**

$$\exists x \exists y(\neg(x = y) \land x \subseteq y \land P_a(x) \land P_a(y))$$

4. The letter $a$ occurs twice not on the same path.

**Solution**

$$\exists x \exists y(\neg(x \subseteq y) \land \neg(y \subseteq x) \land P_a(x) \land P_a(y))$$

5. There exists a subtree with only $a$ letters.

**Solution**

$$\exists x \forall y(x \subseteq y \Rightarrow P_a(y))$$

## Exercise 2 - The limit of WSkS

Prove that the predicate $x = 1y$ is not definable in WSkS.

**Solution**

Assume $x = 1y$ is definable in WSkS. Then there exist a tree automaton recognizing it.

Consider the family of positions $12^n$. In the tree describing $x = 12^n$ and $y = 2^n$, $x$ goes left from the root then only right, and $y$ goes only right from the root. This tree is in the language of the WSkS formula.

Using a refined version of the pumping lemma, as for example in the Exercise session 1 correction, we can increase the size of one of those two branches to reach a contradiction.

## Exercise 3 - The power of WSkS

Produce formulae of WSkS for the following predicates:

1. The set $X$ has exactly two elements.

   **Solution**

   $$\forall Y. Y \subseteq X \Rightarrow (Y = \emptyset \vee \mathrm{Sing}(Y) \vee Y = X)$$

   where $\mathrm{Sing}(Y)$ denotes singletons:

   $$\mathrm{Sing}(X) \stackrel{\text{def}}{=} X \neq \emptyset \wedge \forall Y (Y \subseteq X \Rightarrow (Y = X \vee Y = \emptyset))$$

2. The set $X$ contains at least one string beginning with a 1.

   **Solution**

   $$\exists x. x \in X \wedge 1 \leq x$$

3. Given a formulae of WSkS $\varphi$ with one free first-order variable, produce a formula of WSkS expressing that there is an infinity of words on $\{1, ..., k\}^*$ satisfying $\varphi$.

   **Solution**

   Define $X \vDash \varphi$ by $\forall x, x \in X \Rightarrow \varphi(x)$. Now the wanted formula:

   $$\forall X, X \vDash \varphi \Rightarrow \exists Y, X \subset Y \wedge Y \vDash \varphi$$

# Homework - From formulae to automata

Give tree automata recognizing the languages on trees of maximum arity 2 defined by the following formulae:

1. $(x \in S \wedge (x1 = y \Rightarrow y \in S)) \wedge \left(z \in S \Rightarrow P_f(z)\right)$
2. $\exists S \left[(x \in S \wedge (x1 = y \Rightarrow y \in S)) \wedge \left(z \in S \Rightarrow P_f(z)\right)\right]$