Tree Automata and Applications

Exercise session 5 Solutions

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Exercise 1 - To the infinity...

Let $\Sigma = \{a, b\}$. Define a DFHA $\mathcal{A}$ such that $L(\mathcal{A})$ is the set of all trees such that

For every leaf labeled with a, there is an ancestor from which there is a path whose nodes are labeled with b.

Here, *ancestor* means strict ancestor, and *from which there is a path* means that there is a path from a son of this ancestor to a leaf.

Solution automaton

$Q = \{q_a, q_b, q_\bot\}$, $F = \{q_b, q_\top\}$. The state q_a describes a position with a subtree labelled with an a that doesn't satisfy the wanted property. The state q_b describes a position from which there is a path full of b.

$$a(\varepsilon) \longrightarrow q_a \qquad b(\varepsilon) \longrightarrow q_b$$
$$a(q_\top^+) \longrightarrow q_\top \qquad b(q_\top^+) \longrightarrow q_\top$$
$$a((q_\top \mid q_a)^* q_a (q_\top \mid q_a)^*) \longrightarrow q_a \qquad b((q_\top \mid q_a)^* q_a (q_\top \mid q_a)^*) \longrightarrow q_a$$
$$a(Q^* q_b Q^*) \longrightarrow q_\top \qquad b(Q^* q_b Q^*) \longrightarrow q_b$$

Exercise 2 - PDL

Definition (PDL) The syntax of PDL is the following:

$$\varphi := a \mid \top \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\pi\rangle\varphi \qquad \text{(position formulae)}$$
$$\pi := \downarrow \mid \rightarrow \mid \leftarrow \mid \pi^{-1} \mid \pi;\pi \mid \pi + \pi \mid \pi^* \mid \varphi? \qquad \text{(path formulae)}$$

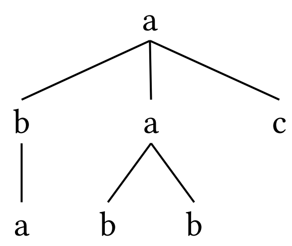
Let t be a tree. We define the semantic $[\![\varphi]\!]_t$ (resp. $[\![\pi]\!]_t$) as a set of positions of t (resp. a relation on positions of t) by induction on the size of φ (resp. π).

$$[\![a]\!]_t = \{w \in \mathrm{Pos}(t) \mid t(w) = a\} \qquad [\![\downarrow]\!]_t = \{(w, w.i) \mid w, w.i \in \mathrm{Pos}(t)\}$$
$$[\![\top]\!]_t = \mathrm{Pos}(t) \qquad [\![\rightarrow]\!]_t = \{(w.i, w.(i+1)) \mid w.i, w.(i+1) \in \mathrm{Pos}(t)\}$$
$$[\![\leftarrow]\!]_t = \{(w.(i+1), w.i) \mid w.i, w.(i+1) \in \mathrm{Pos}(t)\}$$
$$[\![\neg\varphi]\!]_t = \mathrm{Pos}(t) \setminus [\![\varphi]\!]_t \qquad [\![\pi^{-1}]\!]_t = [\![\pi]\!]_t^{-1}$$
$$[\![\varphi_1 \vee \varphi_2]\!]_t = [\![\varphi_1]\!]_t \cup [\![\varphi_2]\!]_t \qquad [\![\pi_1;\pi_2]\!]_t = [\![\pi_2]\!]_t \circ [\![\pi_1]\!]_t$$
$$[\![\langle\pi\rangle\varphi]\!]_t = [\![\pi]\!]_t^{-1}([\![\varphi]\!]_t) \qquad [\![\pi_1 + \pi_2]\!]_t = [\![\pi_2]\!]_t \cup [\![\pi_1]\!]_t$$
$$[\![\pi^*]\!]_t = [\![\pi]\!]_t^* \qquad [\![\varphi?]\!]_t = \Delta_{[\![\varphi]\!]_t} = \{(w, w) \mid w \in [\![\varphi]\!]_t\}$$

Let t be a tree and $w, w' \in \mathrm{Pos}(t)$. We write:

- $t, w \vDash \varphi$ if $w \in [\![\varphi]\!]_t$
- $t \vDash \varphi$ if $t, \varepsilon \vDash \varphi$ and we say that t satisfies φ
- $t, w, w' \vDash \pi$ if $(w, w') \in [\![\pi]\!]_t$

Let t be the tree:



Which formulae are satisfied by t?

1. $\varphi_1 = \neg a \vee \langle\downarrow\rangle\big(\neg\langle\leftarrow\rangle\top \wedge b \wedge \langle\rightarrow^*\rangle(c \wedge \neg\langle\rightarrow\rangle\top)\big)$
2. $\varphi_2 = \neg a \vee \langle\downarrow\rangle\big(\neg\langle\leftarrow\rangle\top \wedge b \wedge \langle(\rightarrow; c?)^*\rangle(\neg\langle\rightarrow\rangle\top)\big)$
3. $\varphi_3 = \langle(a?;\downarrow)^*\rangle(a \wedge \neg\langle\downarrow\rangle\top)$

Solution

1. Yes. The root has a leftmost child which is a b, and which has a rightmost sibling labelled with a c.
2. No. The b child of the root can not go to its rightmost sibling by only going through c nodes.
3. No. There is no path from the root to a leaf with only a.

Exercise 3 - The power of PDL?

Give a translation of PDL in MSO that preserves models. That is, given a position formula φ (resp. a path formula π), construct a MSO formula $\tilde{\varphi}$ (resp. $\tilde{\pi}$) whose set of free variable is $\{X_a \mid a \in \mathcal{F}\} \cup \{x\}$ (resp. $\{X_a \mid a \in \mathcal{F}\} \cup \{x, y\}$) such that

$$t, w \vDash \varphi \text{ iff } (P_a(t))_{a \in \mathcal{F}}, w \vDash \tilde{\varphi}$$

$$\text{resp. } t, w, w' \vDash \pi \text{ iff } (P_a(t))_{a \in \mathcal{F}}, w, w' \vDash \tilde{\pi}$$

where $P_a(t) = \{w \in \mathrm{Pos}(t) \mid t(w) = a\}$.

Solution

Induction on the formula.

$\varphi = a$: $\tilde{\varphi} = x \in X_a$

$\varphi = \varphi_1 \vee \varphi_2$: $\tilde{\varphi} = \tilde{\varphi}_1 \vee \tilde{\varphi}_2$, and same construction for $\neg$ and $\top$.

$\varphi = \langle \pi \rangle \varphi'$: $\tilde{\varphi}\left((X_a)_{a\in\mathcal{F}}, x\right) = \exists y. \tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) \wedge \tilde{\varphi}'\left((X_a)_{a\in\mathcal{F}}, y\right)$

$\pi = \downarrow$: $\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) = \mathrm{Child}(x, y)$

$\pi = \rightarrow$: $\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) = \mathrm{NextSibling}(x, y)$

$\pi = \leftarrow$: $\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) = \mathrm{NextSibling}(y, x)$

$\pi = \pi_0^{-1}$: $\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) = \tilde{\pi}_0\left((X_a)_{a\in\mathcal{F}}, x, y\right)$

$\pi = \pi_1 ; \pi_2$: $\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) = \exists z. \tilde{\pi}_1\left((X_a)_{a\in\mathcal{F}}, x, z\right) \wedge \tilde{\pi}_2\left((X_a)_{a\in\mathcal{F}}, z, y\right)$

$\pi = \pi_1 + \pi_2$: $\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) = \tilde{\pi}_1\left((X_a)_{a\in\mathcal{F}}, x, y\right) \vee \tilde{\pi}_2\left((X_a)_{a\in\mathcal{F}}, x, y\right)$

$\pi = \pi_0^*$:

$$\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) =$$

$$\forall X \left(x \in X \wedge \forall z_1, z_2. \left(\left(z_1 \in X \wedge \tilde{\pi}_0\left((X_a)_{a\in\mathcal{F}}, z_1, z_2\right)\right) \Rightarrow z_2 \in X\right)\right) \Rightarrow y \in X$$

$\pi = ?\,\varphi$: $\tilde{\pi}\left((X_a)_{a\in\mathcal{F}}, x, y\right) = (x = y) \wedge \tilde{\varphi}\left((X_a)_{a\in\mathcal{F}}, x\right)$

Exercise 4 - Propositional Linear-time Temporal Logic

The logic **PTL** is defined as follows:

Syntax P is a finite set of *propositional variables*. Each symbol of P is an atomic formula. If φ and ψ are formulae, then the following also are:

$$\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi, \varphi \mathbf{U} \psi, \mathbf{N}\varphi, \mathbf{L}\varphi$$

Semantics Let P^* be the set of words over the alphabet P. A word $w \in P^*$ is identified with the sequence of letters $w(0)w(1)...w(|w|-1)$. $w(i..j)$ is the word $w(i)...w(j)$. The satisfaction relation is defined by:

- If $p \in P$, $w \vDash p$ if and only if $w(0) = p$.
- The interpretation of logical connectives is the usual one.
- $w \vDash \mathbf{N}\varphi$ if and only if $|w| \geq 2$ and $w(1..|w|-1) \vDash \varphi$.
- $w \vDash \mathbf{L}\varphi$ if and only if $|w| = 1$ and $w \vDash \varphi$.
- $w \vDash \varphi \mathbf{U} \psi$ if and only if there is an index $i \in [\![0, |w|-1]\!]$ such that for all $j \in [\![0, i-1]\!]$, $w(j..|w|-1) \vDash \varphi$ and $w(i..|w|-1) \vDash \psi$.

Let us recall that the language defined by a formula φ is the set of words w sich that $w \vDash \varphi$.

1. What is the language defined by $\mathbf{N}(p_1 \mathbf{U} p_2)$ (with $p_1, p_2 \in P$?)

Solution

It is $L = Pp_1^*p_2P^*$.

Proof that L is included in the langage of the formula
Let $w = lw' \in L$. To prove that $w' \vDash p_1\mathbf{U}p_2$, consider the position of p_2 to satisfy the Until part of the formula.

Proof that the langage of the formula is included in L
The position that satisfies the Until part of the formula is a position with a p_2, and every letter before is a p_1, except for the first one that can be any letter.

2. Give **PTL** formulae defining respectively $P^*p_1P^*, p_1^*, (p_1p_2)^*$.

Solution

Here $\top$ is a shorthand for $p_1 \to p_1$.

$$\top\mathbf{U}p_1$$

$$p_1\mathbf{U}(\mathbf{L}p_1)$$

$$((p_1 \to \mathbf{N}p_2) \land (\mathbf{L}p_2 \lor (p_2 \to \mathbf{N}p_1)))\mathbf{U}(\mathbf{L}p_2)$$

Homework - PTL to WS1S

Give a first-order WS1S formula (i.e. without second-order quantification) that defines the same language as $\mathbf{N}(p_1\mathbf{U}p_2)$.