Tree Automata and Applications

Exercise session 5

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Exercise 1 - To the infinity...

Let $\Sigma = \{a, b\}$. Define a DFHA $\mathcal{A}$ such that $L(\mathcal{A})$ is the set of all trees such that

For every leaf labeled with a, there is an ancestor from which there is a path whose nodes are labeled with b.

Here, *ancestor* means strict ancestor, and *from which there is a path* means that there is a path from a son of this ancestor to a leaf.

Exercise 2 - PDL

Definition (PDL) The syntax of PDL is the following:

$$\varphi := a \mid \top \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\pi\rangle\varphi \qquad \text{(position formulae)}$$

$$\pi := \downarrow \mid \rightarrow \mid \leftarrow \mid \pi^{-1} \mid \pi;\pi \mid \pi + \pi \mid \pi^* \mid \varphi? \qquad \text{(path formulae)}$$

Let t be a tree. We define the semantic $[\![\varphi]\!]_t$ (resp. $[\![\pi]\!]_t$) as a set of positions of t (resp. a relation on positions of t) by induction on the size of φ (resp. π).

$$[\![a]\!]_t = \{w \in \mathrm{Pos}(t) \mid t(w) = a\} \qquad [\![\downarrow]\!]_t = \{(w, w.i) \mid w, w.i \in \mathrm{Pos}(t)\}$$

$$[\![\top]\!]_t = \mathrm{Pos}(t) \qquad [\![\rightarrow]\!]_t = \{(w.i, w.(i+1)) \mid w.i, w.(i+1) \in \mathrm{Pos}(t)\}$$

$$[\![\leftarrow]\!]_t = \{(w.(i+1), w.i) \mid w.i, w.(i+1) \in \mathrm{Pos}(t)\}$$

$$[\![\neg\varphi]\!]_t = \mathrm{Pos}(t) \setminus [\![\varphi]\!]_t \qquad [\![\pi^{-1}]\!]_t = [\![\pi]\!]_t^{-1}$$

$$[\![\varphi_1 \vee \varphi_2]\!]_t = [\![\varphi_1]\!]_t \cup [\![\varphi_2]\!]_t \qquad [\![\pi_1;\pi_2]\!]_t = [\![\pi_2]\!]_t \circ [\![\pi_1]\!]_t$$

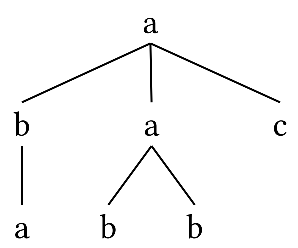
$$[\![\langle\pi\rangle\varphi]\!]_t = [\![\pi]\!]_t^{-1}([\![\varphi]\!]_t) \qquad [\![\pi_1 + \pi_2]\!]_t = [\![\pi_2]\!]_t \cup [\![\pi_1]\!]_t$$

$$[\![\pi^*]\!]_t = [\![\pi]\!]_t^* \qquad [\![\varphi?]\!]_t = \Delta_{[\![\varphi]\!]_t} = \{(w, w) \mid w \in [\![\varphi]\!]_t\}$$

Let t be a tree and $w, w' \in \mathrm{Pos}(t)$. We write:

- $t, w \vDash \varphi$ if $w \in [\![\varphi]\!]_t$
- $t \vDash \varphi$ if $t, \varepsilon \vDash \varphi$ and we say that t satisfies φ
- $t, w, w' \vDash \pi$ if $(w, w') \in [\![\pi]\!]_t$

Let t be the tree:



Which formulae are satisfied by t?

1. $\varphi_1 = \neg a \vee \langle\downarrow\rangle\big(\neg\langle\leftarrow\rangle\top \wedge b \wedge \langle\rightarrow^*\rangle(c \wedge \neg\langle\rightarrow\rangle\top)\big)$
2. $\varphi_2 = \neg a \vee \langle\downarrow\rangle\big(\neg\langle\leftarrow\rangle\top \wedge b \wedge \langle(\rightarrow; c?)^*\rangle(\neg\langle\rightarrow\rangle\top)\big)$
3. $\varphi_3 = \langle(a?;\downarrow)^*\rangle(a \wedge \neg\langle\downarrow\rangle\top)$

Exercise 3 - The power of PDL?

Give a translation of PDL in MSO that preserves models. That is, given a position formula φ (resp. a path formula π), construct a MSO formula $\tilde{\varphi}$ (resp. $\tilde{\pi}$) whose set of free variable is $\{X_a \mid a \in \mathcal{F}\} \cup \{x\}$ (resp. $\{X_a \mid a \in \mathcal{F}\} \cup \{x, y\}$) such that

$$t, w \vDash \varphi \text{ iff } (P_a(t))_{a \in \mathcal{F}}, w \vDash \tilde{\varphi}$$

$$\text{resp. } t, w, w' \vDash \pi \text{ iff } (P_a(t))_{a \in \mathcal{F}}, w, w' \vDash \tilde{\pi}$$

where $P_a(t) = \{w \in \text{Pos}(t) \mid t(w) = a\}$.

Exercise 4 - Propositional Linear-time Temporal Logic

The logic **PTL** is defined as follows:

Syntax P is a finite set of *propositional variables*. Each symbol of P is an atomic formula. If φ and ψ are formulae, then the following also are:

$$\neg\varphi, \varphi \land \psi, \varphi \lor \psi, \varphi \to \psi, \varphi \mathbf{U} \psi, \mathbf{N}\varphi, \mathbf{L}\varphi$$

Semantics Let P^* be the set of words over the alphabet P. A word $w \in P^*$ is identified with the sequence of letters $w(0)w(1)...w(|w|-1)$. $w(i..j)$ is the word $w(i)...w(j)$. The satisfaction relation is defined by:

- If $p \in P$, $w \vDash p$ if and only if $w(0) = p$.
- The interpretation of logical connectives is the usual one.
- $w \vDash \mathbf{N}\varphi$ if and only if $|w| \geq 2$ and $w(1..|w|-1) \vDash \varphi$.
- $w \vDash \mathbf{L}\varphi$ if and only if $|w| = 1$ and $w \vDash \varphi$.
- $w \vDash \varphi \mathbf{U} \psi$ if and only if there is an index $i \in [\![0, |w|-1]\!]$ such that for all $j \in [\![0, i-1]\!]$, $w(j..|w|-1) \vDash \varphi$ and $w(i..|w|-1) \vDash \psi$.

Let us recall that the language defined by a formula φ is the set of words w sich that $w \vDash \varphi$.

1. What is the language defined by $\mathbf{N}(p_1 \mathbf{U} p_2)$ (with $p_1, p_2 \in P$?)
2. Give **PTL** formulae defining respectively $P^* p_1 P^*, p_1^*, (p_1 p_2)^*$.

Homework - PTL to WS1S

Give a first-order WS1S formula (i.e. without second-order quantification) that defines the same language as $\mathbf{N}(p_1 \mathbf{U} p_2)$.