TD 2: LTL Expressivity

Exercise 1 (First order equivalence)**.** Let SU be the temporal operator such that

$$\mu, i \vDash \varphi_1 \,\mathsf{SU}\, \varphi_2 \;\text{ iff }\; \exists k \; i < k \text{ and } \mu, k \vDash \varphi_2 \text{ and } \forall j \in \mathbb{N}, i < j < k \to \mu, j \vDash \varphi_1$$

1 Show that for any LTL(AP, SU) formula φ , there exists an equivalent FO($AP, <$) formula $\overline{\varphi}$ namely $\overline{\varphi}$ satisfiable iff φ is. Try using only 3 variables.

2. Express the operators X and U using only $\mathsf{SU}, \vee, \neg, \top$

3. Show that for any LTL(AP, X, U) formula, there exists an equivalent FO($AP, <$) formula using only 3 variables.

Exercise 2 (Game)**.** Consider the game $\mathcal{G}(M, M')$ with M and M' being linear temporal structures with labeling in 2^{AP}. A configuration corresponds to a pair (x, x') with x being a position in M et x a position in M'. A configuration is said *compatible* iff the labeling of x and x' are the same.
On each turn, the player 0 (P_0) chooses either M or M' and changes the associated position x (resp. x') to a y (resp. y') s.t. $y \geq x$ (resp. $y' \geq x'$). The player 1 (P_1) must then change the position on the other structure so the new configuration is compatible.
P_0 wins the game $\mathcal{G}_r(M, M')$ if P_1 can not play anymore, and $P1$ wins if they can respond a given number r of turns.
Let LTL$_r(F, G)$ be the fragment of LTL(F, G) with formulas of temporal height at most r.
We want to show by induction on r that P_1 has a winning strategy in $\mathcal{G}_r(M, M')$ iff $M, 0$ and $M', 0$ satisfy the same formulas of LTL$_r(F, G)$.

1. Show the result for $r = 1$

2. Show the direct implication using an induction on a formula φ satisfied by $M, 0$.

3. Show the inverse implication by contraposition.

Homework

To hand in on October 10th at the beginning of the exercise session, or at any earlier time by email at `nicolas.dumange@ens-paris-saclay.fr`. You can use either French or English.

Exercise 3 (Gabbay separation)**.** Express the following LTL($AP, \mathsf{Y}, \mathsf{S}, \mathsf{X}, \mathsf{U}$) formula as an equivalent boolean combination of pure future (LTL($AP, \mathsf{X}, \mathsf{U}$)) and pure past (LTL($AP, \mathsf{Y}, \mathsf{S}$)) formulas.

- $\top$

1. $a \,\mathsf{U}\, Yb$
2. $\mathsf{G}(a \,\mathsf{S}\, (\neg \mathsf{X}\, b))$
3. $\mathsf{F}(\mathsf{P}(a))$
4. $\mathsf{G}\,\mathsf{F}(a \,\mathsf{S}\, b)$