TD 2: Temporal Logics
Solutions

Exercice 1

1. Using the definition of SU, one can identify a formula φ with some FO formula $\overline{\varphi}$ having for only one free variable, x, and denoted $\overline{\varphi}(x)$.

   $\overline{\varphi}(x)$ is defined inductively on the structure of φ:

   * For $\varphi = a$ with $a \in AP$, $\overline{a}(x) = P_a(x)$
   * For $\varphi = \varphi_1 \mathsf{SU} \varphi_2$, $\overline{\varphi}(x) = \exists z : x < z \wedge \overline{\varphi_1}(z) \wedge \forall y : x < y < z \to \overline{\varphi_2}(y)$. By induction hypothesis, $\overline{\varphi_1}$ and $\overline{\varphi_2}$ only use 3 variables, two of them being bound in the formula, and thus can be named among x, y and z, and the other one being bound in $\overline{\varphi}$.

2. * For φ an LTL formula,

   $$\begin{aligned} X\varphi &\equiv \bot \mathsf{SU} \varphi \\ &\equiv (\neg\top) \mathsf{SU} \varphi \end{aligned}$$

   * For φ_1 and φ_2 LTL formulas,

   $$\varphi_1 \mathsf{U} \varphi_2 \equiv \varphi_2 \vee (\varphi_1 \wedge (\varphi_1 \mathsf{SU} \varphi_2))$$

3. According to question 2, for any formula in LTL$(AP, \mathsf{X}, \mathsf{U})$, there exists an equivalent formula using for only temporal operator SU. Thus, using question 1, for any LTL$(AP, \mathsf{X}, \mathsf{U})$, there exists an equivalent formula in FO$(<)$ using only 3 variables.

Exercise 2

Additional assumption: $(0, 0)$ is compatible

1. Suppose P_1 has a winning strategy. Let φ be a formula in LTL$_1(\mathsf{F}, \mathsf{G})$. We show by induction on the structure of φ that $M, 0 \models \varphi$ iff $M', 0 \models \varphi$. By symmetry of the roles of M and M', we only have to prove that $M, 0 \models \varphi$ implies $M', 0 \models \varphi$.

   - $\varphi = \mathsf{F}\varphi'$. φ' being of temporal height 0, it is a boolean combination of atomic proposition, and since $(0, 0)$ is compatible, it holds that $M, 0 \models \varphi'$ iff $M', 0 \models \varphi'$. Since $M, 0 \models \mathsf{F}\varphi'$, there exists $i \in \mathbb{N}$ such that $M, i \models \varphi'$. P_1 having winning strategy, there also exists a j such that $M', j \models \varphi'$, and $M', 0 \models \mathsf{F}\varphi'$.
   - $\varphi = \mathsf{G}\varphi'$. Suppose by contradiction that $M', 0 \not\models \mathsf{G}\varphi'$. Thus there exists $j \in \mathbb{N}$ such that $M', j \not\models \varphi$. Thus, if P_0 plays j on M', since P_1 has a winning strategy, there exists $i \in \mathbb{N}$ such that P_1 can play on M and $M, i \models \neg\varphi'$, which contradicts $M, 0 \models \mathsf{G}\varphi'$, and $M', 0 \models G\varphi'$.

- $\varphi = \varphi_1 \vee \varphi_2$ (resp. $\varphi = \neg\varphi_1$).
  By induction hypothesis, $M, 0 \models \varphi_\iota$ iff $M', 0 \models \varphi_\iota$, for $\iota \in \{1, 2\}$.
  Yet, since $M, 0 \models \varphi_1 \vee \varphi_2$ holds, either $M, 0 \models \varphi_1$ or $M, 0 \models \varphi_2$ (resp. $M, 0 \not\models \varphi_1$) holds. Thus, either $M', 0 \models \varphi_1$ or $M', 0 \models \varphi_2$ (resp. $M', 0 \not\models \varphi_1$) holds. Hence $M', 0 \models \varphi$.

Reciprocally, suppose $M, 0$ and $M', 0$ satisfy the same $\mathrm{LTL}_1(\mathsf{F}, \mathsf{G})$ formulas. Then suppose P_0 plays x on M. Since $M, 0 \models \mathsf{F}\, M[x]$ holds, $M', 0 \models M[x]$ also holds, and there exists a winning move for any move of P_0. Thus, there exists a winning strategy for P_1 in $\mathcal{G}_1(M, M')$.

2. We introduce M_ι and M'_ι the sequences M and M' shifted by ι ranks, namely $M_\iota[i] = M[i + \iota]$. We consider the induction hypothesis that for all $r' < r$, and for all M, M', there exists a winning strategy for P_1 in $\mathcal{G}'_r(M, M')$ iff $M, 0$ and $M', 0$ satisfy the same $\mathrm{LTL}_{r'}$ formulas.

   Suppose there exists a winning strategy for P_1 in $\mathcal{G}_r(M, M')$. Thus, for any action x from P_0, there exists an action x' from P_1 such that there exists a winning strategy for P_1 in $\mathcal{G}_{r-1}(M_x, M'_{x'})$. Let φ be a formula in $\mathrm{LTL}_1(\mathsf{F}, \mathsf{G})$. We show by induction on the structure of φ that $M, 0 \models \varphi$ iff $M', 0 \models \varphi$.

   - $\varphi = \mathsf{F}\,\varphi'$. Suppose $M, 0 \models \mathsf{F}\,\varphi'$. Thus, there exists $i \in \mathbb{N}$ such that $M, i \models \varphi'$. Suppose by contradiction there is no j such that $M', j \models \varphi'$. Thus, for all j, P_1 does not have a winning strategy in $\mathcal{G}_{r-1}(M_i, M'_j)$, by induction hypothesis, and there is no move j for P_1 to counter the move i for P_0, hence the contradiction. Thus there exists j such that $M', j \models \varphi'$, and $M', 0 \models \varphi$.
   - $\varphi = \mathsf{G}\,\varphi'$. Suppose $M, 0 \models \mathsf{G}\,\varphi'$, and suppose by contradiction that $M', 0 \not\models \mathsf{G}\,\varphi'$ does not hold. Then there exists j such that $M', j \not\models \varphi'$. Yet, P_1 can react to P_0 playing j, thus there exists i such that P_1 have a winning strategy in $\mathcal{G}_{r-1}(M_i, M'_j)$, and by induction hypothesis, $M, i \not\models \varphi'$, hence the contradiction. Thus $M', 0 \models \varphi$.
   - The remaining cases are the same as wit $r = 1$.

3. Suppose $M, 0$ and $M', 0$ satisfy the same $\mathrm{LTL}_r(\mathsf{F}, \mathsf{G})$ formula. Consider by contradiction that P_0 plays x on M and P_1 can not surely win. Then, for all x', it holds that P_1 does not have a winning strategy in $\mathcal{G}_{r-1}(M_x, M'_{x'})$. Thus, by induction hypothesis, for all x' there exists a formula $\varphi_{x'}$ in $\mathrm{LTL}_r(\mathsf{F}, \mathsf{G})$ such that $M, x \models \not\varphi_{x'}$ and $M', x' \models \varphi_{x'}$. Yet, there is a finite number of formula in $\mathrm{LTL}_r(\mathsf{F}, \mathsf{G})$, up to equivalence. Thus, we can consider a unique representative of each class among the $\varphi_{x'}$, and there is a finite formula $\Psi = \mathsf{G} \bigvee_{x' \in \mathbb{N}} \varphi_{x'}$ such that $M, 0 \not\models \Psi$ and $M', 0 \models \Psi$, hence the contradiction.

   P_1 has a winning strategy in $\mathcal{G}_r(M, M')$ iff $M, 0$ and $M', 0$ satisfy the same formulas.