Weighted versus Probabilistic Logics

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Motivations

Analysis of quantitative systems

- Probabilistic Systems
- Minimization of costs
- Maximization of rewards
- Computation of reliability
- Optimization of energy consumption
- ...

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Models

- Probabilistic automata (generative, reactive)
- Transition systems with costs or rewards
- ...

All are special cases of Weighted Automata.

Motivations

Specification

PCTL: Probabilistic CTL	Hansson & Jonsson, '94
PCTL*: Probabilistic CTL*	de Alfaro, '98
CTL\$: Valued CTL	Buchholz & Kemper, '03, '09
wMSO: Weighted MSO	Droste & Gastin, '05, '07, '09

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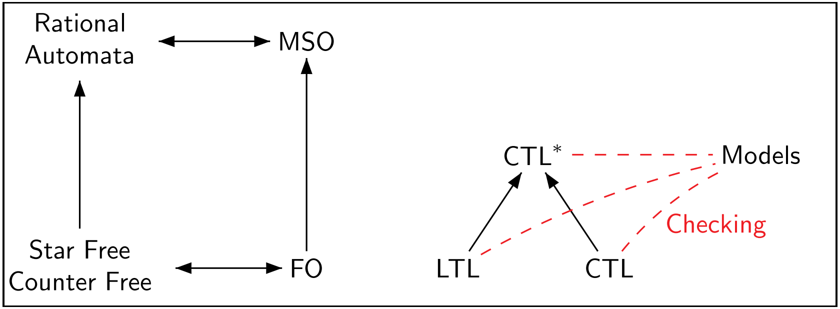
Natural Problems

- Satisfiability
- Model Checking
- Expressivity

Qualitative (Boolean) Picture

$$\begin{tikzcd}
\text{Rational} \\ \text{Automata} \arrow[r, leftrightarrow] & \text{MSO} \\
\text{Star Free} \\ \text{Counter Free} \arrow[u] \arrow[r, leftrightarrow] & \text{FO} \arrow[u]
\end{tikzcd}$$

Qualitative (Boolean) Picture



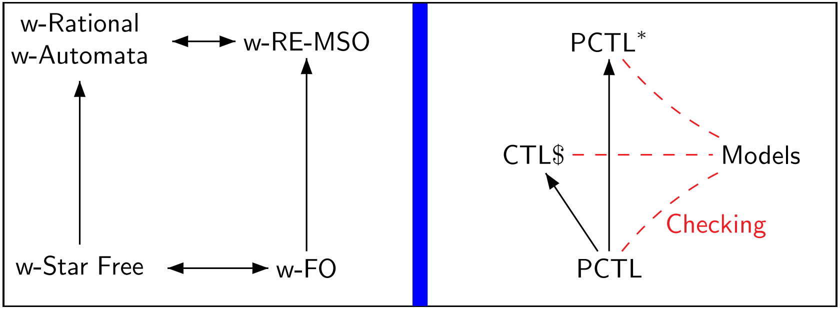
Qualitative (Boolean) Picture

$$\begin{tikzcd}
\begin{array}{c}\text{Rational}\\\text{Automata}\end{array} \arrow[r, leftrightarrow] & \text{MSO} \arrow[rr, leftrightarrow] & & \mu\text{-calculus} & & \\
& & & \text{CTL}^{*} \arrow[u] \arrow[rr, dash, dashed] & & \text{Models} \arrow[dlll, dash, dashed] \arrow[dl, dash, dashed, "\text{Checking}"] \\
\begin{array}{c}\text{Star Free}\\\text{Counter Free}\end{array} \arrow[uu] \arrow[r, leftrightarrow] & \text{FO} \arrow[uu] \arrow[r, leftrightarrow] & \text{LTL} \arrow[ur] & & \text{CTL} \arrow[ul] &
\end{tikzcd}$$

Quantitative Picture

$$\begin{tikzcd}
\substack{\text{w-Rational}\\ \text{w-Automata}} \arrow[r, leftrightarrow] & \text{w-RE-MSO} \\
\text{w-Star Free} \arrow[u] \arrow[r, leftrightarrow] & \text{w-FO} \arrow[u]
\end{tikzcd}$$

Quantitative Picture



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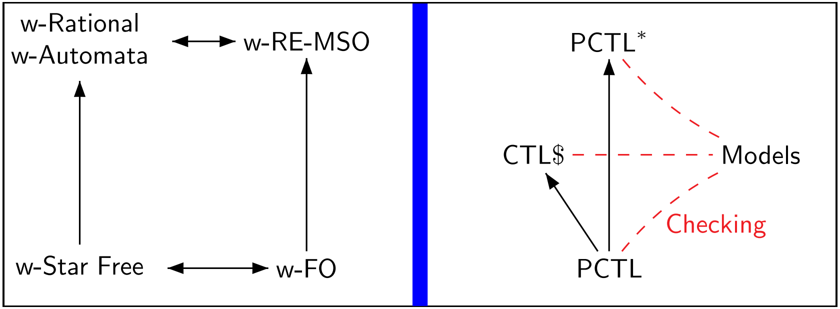


Our aim is to compare and unify these logics

Plan

1. Weighted Automata

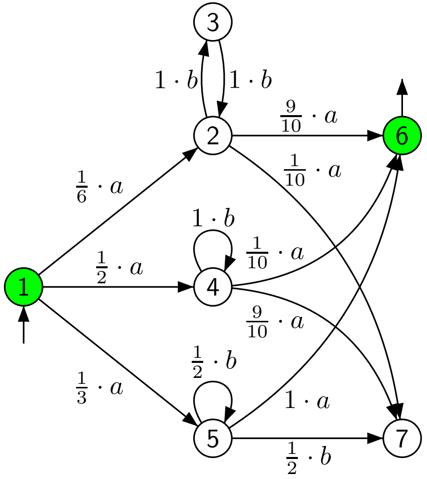
Weighted MSO Logic

Weighted CTL* **and** PCTL*

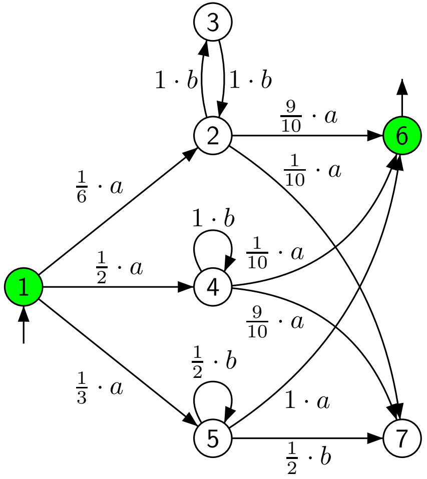
Weighted CTL* **versus weighted** MSO

Conclusion and Open problems

Weighted Automata by Examples



Weighted Automata by Examples



Several paths for $v = ab^n a$:

$\pi_1 = 1 \xrightarrow{a} 4 \xrightarrow{b} 4 \cdots 4 \xrightarrow{b} 4 \xrightarrow{a} 6$
$\text{weight}(\pi_1) = \frac{1}{2} \cdot 1^n \cdot \frac{1}{10} = \frac{1}{20}$

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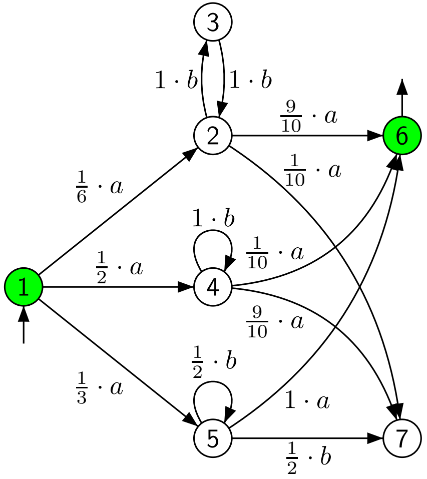

Several paths for $v = ab^na$:

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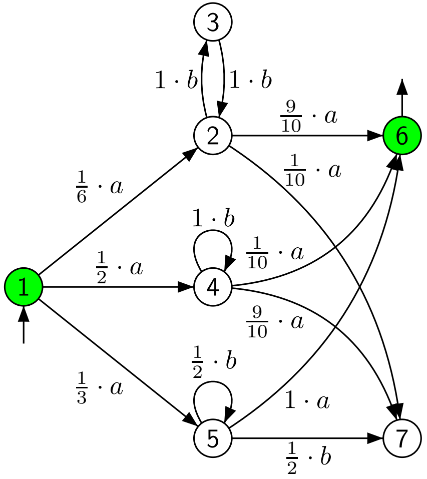
$\pi_2 = 1 \xrightarrow{a} 5 \xrightarrow{b} 5 \cdots 5 \xrightarrow{b} 5 \xrightarrow{a} 6$
$\mathrm{weight}(\pi_2) = \frac{1}{3} \cdot (\frac{1}{2})^n \cdot 1 = \frac{1}{3 \cdot 2^n}$

Weighted Automata by Examples



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If n is even:
$\pi_3 = 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \cdots 2 \xrightarrow{a} 6$
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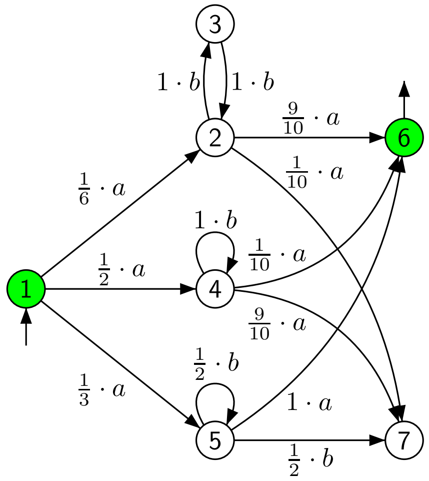
Weighted Automata by Examples



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Probabilistic: $\mathbb{P}\text{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$

$$[\![\mathcal{A}]\!](v) = \begin{cases} \frac{1}{20} + \frac{1}{3 \cdot 2^n} & \text{if } n \text{ is odd} \\ \frac{1}{5} + \frac{1}{3 \cdot 2^n} & \text{if } n \text{ is even} \end{cases}$$

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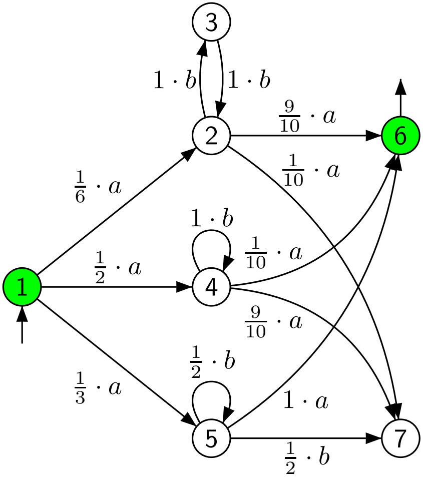

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Reliability: $([0,1], \max, \cdot, 0, 1)$

$$[\![\mathcal{A}]\!](v) = \begin{cases} \max(\frac{1}{20}, \frac{1}{3 \cdot 2^n}) & \text{if } n \text{ is odd} \\ \max(\frac{3}{20}, \frac{1}{3 \cdot 2^n}) & \text{if } n \text{ is even} \end{cases}$$

Semirings

Definition: Semiring

- $\mathbb{K} = (K, \oplus, \otimes, \mathbf{0}, \mathbf{1})$
- $(K, \oplus, \mathbf{0})$ is a commutative monoid,
- $(K, \otimes, \mathbf{1})$ is a monoid,
- multiplication distributes over addition, and $\mathbf{0}$ is absorbant.

Examples:

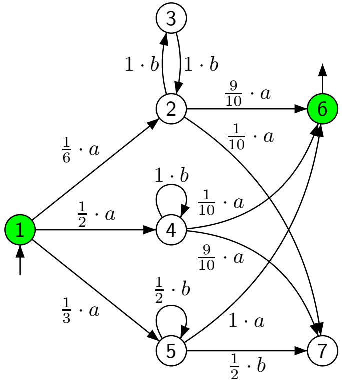
- Boolean: $\mathbb{B} = (\{\mathbf{0}, \mathbf{1}\}, \vee, \wedge, \mathbf{0}, \mathbf{1})$
- Natural: $(\mathbb{N}, +, \cdot, 0, 1)$
- Tropical: $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$
- Probabilistic: $\mathbb{P}\text{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$
- Reliability: $([0, 1], \max, \cdot, 0, 1)$

Reactive Probabilistic Finite Automata

Definition: RPFA on $\mathbb{P}\mathrm{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$

A *reactive probabilistic finite automaton (*RPFA*)* is a weighted automaton $\mathcal{A} = (Q, q_0, \mu, F)$ over $\mathbb{P}\mathrm{rob}$ such that, for all $q \in Q$ and $a \in \Sigma$,

$$\sum_{q' \in Q} \mu(q, a, q') \in \{0, 1\}$$

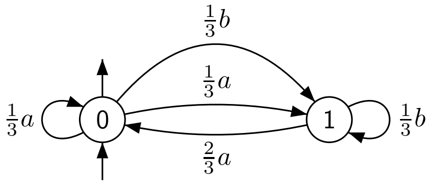


Generative Probabilistic Finite Automata

Definition: $\textsc{gPFA}$ on $\mathbb{P}\text{rob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$

A *generative probabilistic finite automaton (*$\textsc{gPFA}$*)* is a weighted automaton $\mathcal{A} = (Q, q_0, \mu, F)$ over $\mathbb{P}$rob such that, for all $q \in Q$,

$$\sum_{(a,q') \in \Sigma \times Q} \mu(q, a, q') \in \{0, 1\}$$



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Weighted MSO

Short history

Introduced by Droste & Gastin (ICALP'05)
Aim: Logical characterization of weighted automata.
Generalization of Elgot's and Büchi's theorems.

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Trees	Droste & Vogler
Infinite words	Droste & Kuske, Droste & Rahonis
Pictures	Fischtner
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No link with quantitative temporal logics such as PCTL or CTL$.

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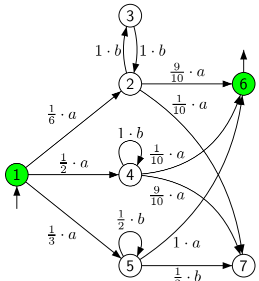
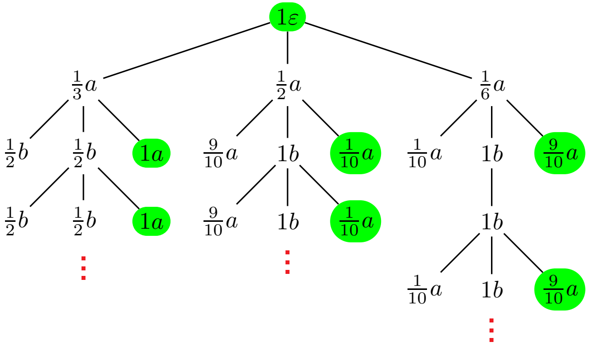
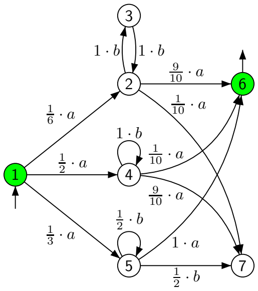
Semantics on unfoldings of weighted automata

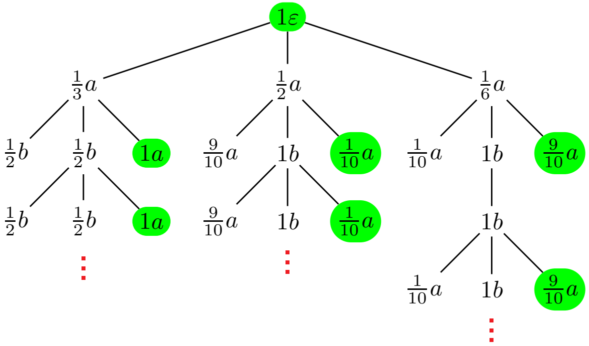
Unfoldings of Weighted Automata





Unfoldings of Weighted Automata





Definition: Weighted Tree

$$\begin{array}{rccc} t: & D^* & \rightharpoonup & K \times \Sigma \\ & u & \rightarrow & (\kappa_t(u), \ell_t(u)) \end{array}$$

Extended Weighted MSO

Definition: Vocabulary

- $\mathcal{C}$ is a vocabulary of symbols $\bowtie \in \mathcal{C}$ with $\mathrm{arity}(\bowtie) \in \mathbb{N}$.
  - $\mathcal{C} = \{\vee, \wedge, \neg\}$
  - $\mathcal{C} = \{\wedge, \neg, <\}$

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  - $\mathcal{C} = \{\vee, \wedge, \neg\}$
  - $\mathcal{C} = \{\wedge, \neg, <\}$
- Each symbol $\bowtie \in \mathcal{C}$ is given a semantics $[\![\bowtie]\!] : K^{\mathrm{arity}(\bowtie)} \rightharpoonup K$.
  - $[\![\vee]\!] = \oplus$
  - $[\![\wedge]\!] = \otimes$
  - $[\![\neg]\!](k) = \begin{cases} \mathbf{1} & \text{if } k = \mathbf{0} \\ \mathbf{0} & \text{otherwise} \end{cases}$
  - Probabilistic: $[\![\neg]\!](k) = 1 - k$ or $[\![\neg]\!](k) = \max(0, 1 - k)$
  - Ordered semiring: $[\![<]\!] : K^2 \to \{\mathbf{0}, \mathbf{1}\}$

Extended Weighted MSO

Definition: Syntax of wMSO$(\mathbb{K}, \Sigma, \mathcal{C})$

$$\begin{aligned}\varphi ::= k \mid \kappa(x) \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)})\\ \mid P_a(x) \mid x \leq y \mid x \in X \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi\end{aligned}$$

where $k \in K$, $a \in \Sigma$, x, y are first-order variables, X is a set variable and $\bowtie \in \mathcal{C}$.

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Let $\mathcal{V}$ be a finite set of first-order and second-order variables with $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$.

Let $t : \begin{array}{ccc} D^* & \rightharpoonup & K \times \Sigma \\ u & \to & (\kappa_t(u), \ell_t(u)) \end{array}$ be a weighted tree and σ a $(\mathcal{V}, t)$-assignment.

$$[\![k]\!]_{\mathcal{V}}(t, \sigma) = k$$

$$[\![\kappa(x)]\!]_{\mathcal{V}}(t, \sigma) = \kappa_t(\sigma(x))$$

$$[\![P_a(x)]\!]_{\mathcal{V}}(t, \sigma) = \begin{cases} \mathbf{1} & \text{if } \ell_t(\sigma(x)) = a \\ \mathbf{0} & \text{otherwise} \end{cases}$$

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$$[\![\bowtie(\varphi_1, \ldots, \varphi_r)]\!]_{\mathcal{V}}(t, \sigma) = [\![\bowtie]\!]([\![\varphi_1]\!]_{\mathcal{V}}(t, \sigma), \ldots, [\![\varphi_r]\!]_{\mathcal{V}}(t, \sigma)) \qquad \text{if } \mathrm{arity}(\bowtie) = r$$

Recall that $[\![\vee]\!] = \oplus$ and $[\![\wedge]\!] = \otimes$

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$$[\![x \leq y]\!]_{\mathcal{V}}(t, \sigma) = \begin{cases} \mathbf{1} & \text{if } \sigma(x) \leq \sigma(y) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$\leq$ is the prefix ordering on $\mathrm{dom}(t)$

$$[\![x \in X]\!]_{\mathcal{V}}(t, \sigma) = \begin{cases} \mathbf{1} & \text{if } \sigma(x) \in \sigma(X) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

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$$[\![\exists x.\varphi]\!]_{\mathcal{V}}(t,\sigma) = \bigoplus_{u \in \mathrm{dom}(t)} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}(t, \sigma[x \to u])$$

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Examples and Macros

Example:

Let $\varphi_1 = \exists x.(P_b(x) \wedge (\kappa(x) > 0))$.

$$[\![\varphi_1]\!](t) = \bigoplus_{u \in \mathrm{dom}(t)} P_b(u) \wedge (\kappa(u) > 0)$$

is the number of nodes labeled b and having a positive weight.

Examples and Macros

Definition: Useful macro

$$\varphi_1 \xrightarrow{+} \varphi_2 \overset{\text{def}}{=} \neg\varphi_1 \vee (\varphi_1 \wedge \varphi_2)$$

If φ_1 is boolean, we have

$$[\![\varphi_1 \xrightarrow{+} \varphi_2]\!]_{\mathcal{V}}(t,\sigma) = \begin{cases} [\![\varphi_2]\!]_{\mathcal{V}}(t,\sigma) & \text{if } [\![\varphi_1]\!]_{\mathcal{V}}(t,\sigma) = \mathbf{1} \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

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Example:

Let $\varphi_2 = \forall x.((P_a(x) \wedge (\kappa(x) > 0)) \xrightarrow{+} \kappa(x))$.

$$[\![\varphi_1]\!](t) = \bigotimes_{u \in \mathrm{dom}(t)} ((P_a(x) \wedge (\kappa(x) > 0)) \xrightarrow{+} \kappa(x))$$

multiplies the positive values of all a-labeled nodes.

Examples and Macros

Definition: Macros for Boolean formulas

$$\varphi_1 \underline{\vee} \varphi_2 \overset{\text{def}}{=} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$$

$$\underline{\exists} x.\varphi \overset{\text{def}}{=} \neg\forall x.\neg\varphi$$

$$\underline{\exists} X.\varphi \overset{\text{def}}{=} \neg\forall X.\neg\varphi$$

Hence, we can easily define boolean formulas for all MSO properties.

Examples and Macros

Definition: Macros for Boolean formulas

$$\varphi_1 \underline{\vee} \varphi_2 \stackrel{\text{def}}{=} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$$
$$\underline{\exists} x.\varphi \stackrel{\text{def}}{=} \neg\forall x.\neg\varphi$$
$$\underline{\exists} X.\varphi \stackrel{\text{def}}{=} \neg\forall X.\neg\varphi$$

Hence, we can easily define boolean formulas for all MSO properties.

Example:

- Let $\mathrm{path}(x, X)$ be a boolean formula stating that X is a maximal path starting from node x,
- The boolean formula $\underline{\exists} z.(z \in X \wedge P_b(z) \wedge \forall y.(x < y < z \xrightarrow{+} P_a(y)))$ checks if X satisfies a SU b,
- The quantitative formula $\xi(x, X) = \forall y.((y \in X \wedge x < y) \xrightarrow{+} \kappa(y))$ computes the weight of path X, i.e., the product of weights of nodes in $X \setminus \{x\}$.

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Then, we compute the sum of weights of paths from x satisfying a SU b with

$$\exists X.(\mathrm{path}(x, X) \wedge \underline{\exists} z.(z \in X \wedge P_b(z) \wedge \forall y.(x < y < z \xrightarrow{+} P_a(y))) \wedge \xi(x, X))$$

Original Weighted MSO

Definition: Original Weighted MSO Droste & Gastin

- $\mathcal{C} = \{\vee, \wedge\}$
- negations of atomic formulas
- $\kappa(x)$ is not allowed

sREMSO is a syntactic restriction of the existential fragment.

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Theorem: Droste & Gastin

- From any w-Aut $\mathcal{A}$ we can construct a formula φ in sREMSO s.t. $[\![\varphi]\!] = [\![\mathcal{A}]\!]$,
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- From any formula φ in sREMSO we can construct a w-Aut $\mathcal{A}$ s.t. $[\![\varphi]\!] = [\![\mathcal{A}]\!]$.

Definition: Satisfiability

A formula φ is satisfiable if $[\![\varphi]\!](t) \neq \mathbf{0}$ for some t.

Corollary: Satisfiability

The satisfiability problem is decidable for sREMSO.

Extended Weighted MSO

Proposition: Satisfiability

The satisfiability problem for wMSO$(\mathbb{P}\mathrm{rob}, \Sigma, \{\vee, \wedge, \neg, <\})$ is undecidable.

Proof:

Let $\mathcal{A} = (Q, q_0, \mu, F)$ be a reactive probabilistic finite automaton over Σ.

By [DG], $\exists \varphi \in \mathrm{sREMSO}(\mathbb{P}\mathrm{rob}, \Sigma, \{\vee, \wedge, \neg\})$ such that $[\![\varphi]\!](w) = [\![\mathcal{A}]\!](w)$ for all unweighted words $w \in \Sigma^*$.

Since φ does not use $\kappa(x)$, considering weighted or unweighted words or trees does not make any difference.

Now, for $p \in [0, 1]$ and $w \in \Sigma^*$ we have $[\![p < \varphi]\!](w) \neq 0$ iff $[\![\mathcal{A}]\!](w) > p$.

Hence, $p < \varphi$ is satisfiable iff the automaton $\mathcal{A}$ with threshold p accepts a nonempty language. By Rabin's Theorem, this is undecidable.

Plan

Weighted CTL*

Definition: Syntax of wCTL$^*(\mathbb{K}, Prop, \mathcal{C})$

Boolean path formulas: $\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi \ \mathsf{SU} \ \psi$

Quantitative state formulas: $\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\text{arity}(\bowtie)}) \mid \mu(\psi)$

where $p \in Prop$, $k \in K$, $\bowtie \in \mathcal{C}$.

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Definition: Semantics for Boolean path formulas

$$\begin{array}{llll} t: & D^* & \rightharpoonup & K \times \Sigma \\ & u & \rightarrow & (\kappa_t(u), \ell_t(u)) \end{array}$$

weighted tree, w branch of t, u node on w.

$t, w, u \models \varphi$ if $[\![\varphi]\!](t,u) \neq \mathbf{0}$

$t, w, u \models \psi_1 \wedge \psi_2$ if $t, w, u \models \psi_1$ and $t, w, u \models \psi_2$

$t, w, u \models \neg\psi$ if $t, w, u \not\models \psi$

$t, w, u \models \psi_1 \text{ SU } \psi_2$ if $\exists u < v \leq w : (t, w, v \models \psi_2$ and $\forall u < v' < v : t, w, v' \models \psi_1)$

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Definition: Semantics for quantitative state formulas

$$\begin{array}{llll} t: & D^* & \rightharpoonup & K \times \Sigma \\ & u & \rightarrow & (\kappa_t(u), \ell_t(u)) \end{array}$$

weighted tree, u node of t, $\Sigma = 2^{Prop}$.

$$[\![k]\!](t,u) = k$$

$$[\![\kappa]\!](t,u) = \kappa_t(u)$$

$$[\![p]\!](t,u) = \begin{cases} \mathbf{1} & \text{if } p \in \ell_t(u) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$$[\![\bowtie(\varphi_1, \ldots, \varphi_r)]\!](t,u) = [\![\bowtie]\!]([\![\varphi_1]\!](t,u), \ldots, [\![\varphi_r]\!](t,u)) \qquad \text{if arity}(\bowtie) = r$$

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$$\begin{array}{rcll} t: & D^* & \rightharpoonup & K \times \Sigma \\ & u & \to & (\kappa_t(u), \ell_t(u)) \end{array}$$

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$$[\![\mu(\psi)]\!](t,u) = \bigoplus_{w \in \text{Branches}(t) \,\mid\, t,w,u \models \psi} \ \bigotimes_{v \,\mid\, u < v \leq w} \kappa_t(v)$$

Example for $\mu(\psi)$

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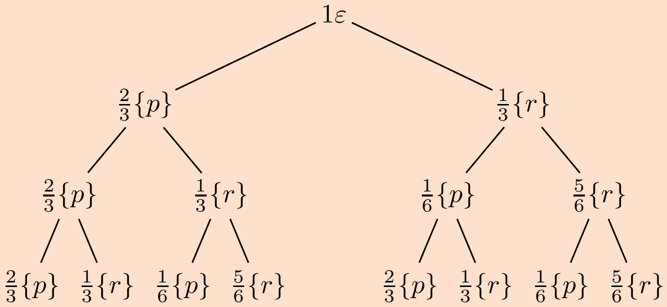

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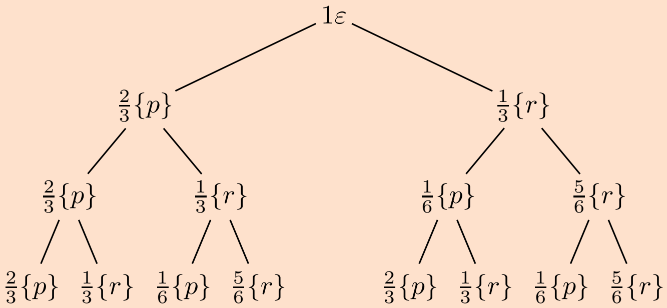
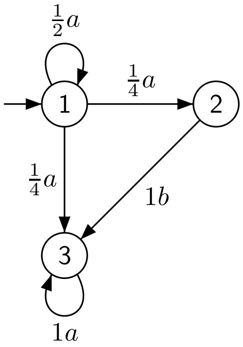
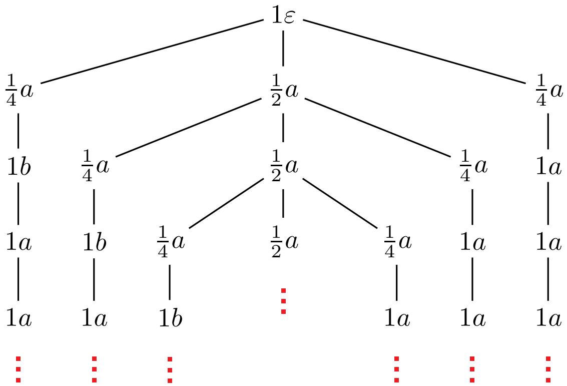


$$[\![\mu(p\,\mathsf{SU}\,r)]\!](t) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}\cdot\left(\frac{1}{6}+\frac{5}{6}\right) + \frac{1}{3}\cdot(1) = \frac{19}{27}$$

Unfoldings are infinite (regular) trees

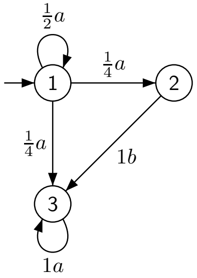


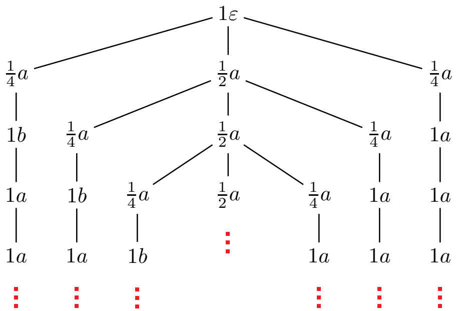


Infinite sums and products

Example:

$$[\![\mu(a^*ba^\omega)]\!](t,\varepsilon) = \bigoplus_{w \text{ left branch}} \bigotimes_{v \mid \varepsilon < v \le w} \kappa_t(v) = \sum_{n\geq 0} \frac{1}{2^n}\cdot\frac{1}{4}\cdot 1^\omega = \frac{1}{2}$$



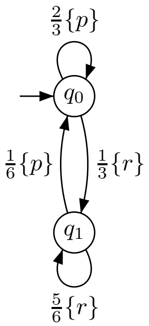


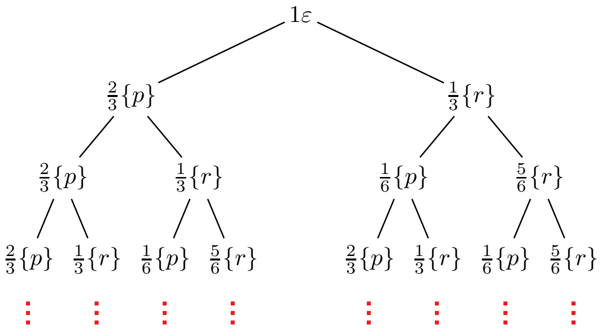
We need infinite sums and products

Some well-defined infinite sums or products

- $\bigoplus_{i \in I} k_i$ is well defined if $|\{i \in I \mid k_i \neq 0\}| < \infty$,
- $\bigotimes_{i \in I} k_i$ is well defined if $|\{i \in I \mid k_i \neq 1\}| < \infty$,
- $\bigotimes_{i \in I} k_i$ is well defined if $k_i = 0$ for some $i \in I$,
- $\sum_{i \geq 0} \frac{1}{2^i}$

Unfoldings of gPFA





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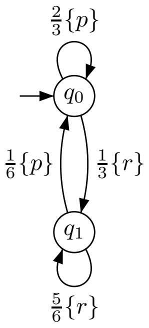

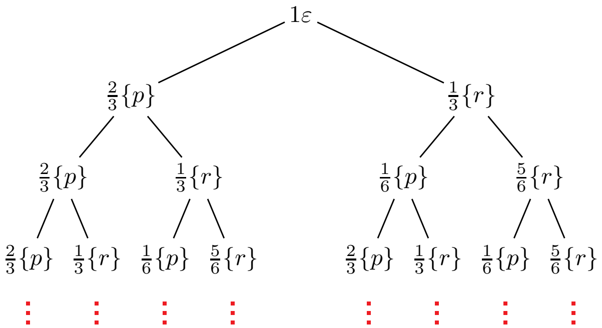

Probability measure

- The weight of each branch is an infinite product which converges to 0.
- The sum of the weights of all branches starting from any node should be 1.
- To define $[\![\mu(\psi)]\!]$, we use the probability measure on the sequence space.

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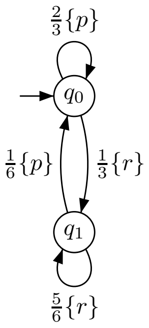

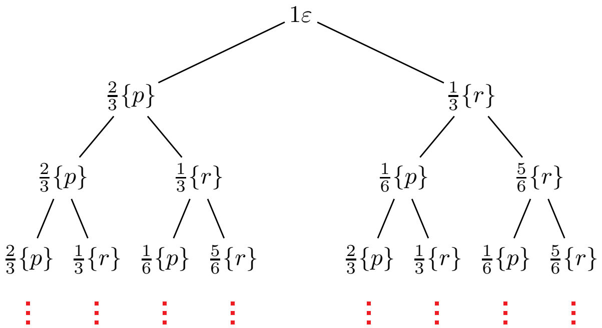

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- To define $[\![\mu(\psi)]\!]$, we use the probability measure on the sequence space.
- We get $[\![\mu(p\,\mathsf{SU}\,r)]\!](t,\varepsilon) = \sum_{n\geq 0}\left(\frac{2}{3}\right)^n \cdot \frac{1}{3} = 1.$

PCTL* **is a boolean fragment of** wCTL*

Definition: Probabilistic computation tree logic PCTL* de Alfaro '98

The syntax of PCTL* is given by:

Boolean path formulas: $\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi \,\mathsf{SU}^{\leq n}\, \psi$

Boolean state formulas: $\varphi ::= 0 \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mu(\psi) \geq k \mid \mu(\psi) > k$

where $n \in \mathbb{N} \cup \{\infty\}$, $p \in \mathit{Prop}$, $k \in [0,1]$.

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Recall: Syntax of wCTL$^*(\mathbb{P}\mathrm{rob}, Prop, \{\neg, \wedge, \ge\})$

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Remark: PCTL* is a boolean fragment of wCTL*

State formulas are restricted:

- do not use κ,
- use $\geq$ and $\mu(\psi)$ only in comparisons of the form: $(\mu(\psi) \geq k)$ and $(k \geq \mu(\psi))$

wCTL **is a fragment of** wCTL*

Definition: Syntax of wCTL$(\mathbb{K}, Prop, \mathcal{C})$

Only quantitative state formulas:

$$\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)}) \mid \mu(\varphi \,\mathsf{SU}^{\leq n}\, \varphi)$$

where $p \in Prop$, $k \in K$, $\bowtie \in \mathcal{C}$, $n \in \mathbb{N} \cup \{\infty\}$.

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Boolean path formulas are restricted to $\psi ::= \varphi \ \mathsf{SU}^{\leq n}\ \varphi$

PCTL **is a fragment of** wCTL

Definition: Probabilistic CTL Hansson & Jonsson '94

Only Boolean state formulas:

$$\varphi ::= 0 \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mu(\varphi \, \mathsf{SU}^{\leq n} \, \varphi) \geq k \mid \mu(\varphi \, \mathsf{SU}^{\leq n} \, \varphi) > k$$

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Remark: PCTL is a fragment of wCTL$(\mathbb{P}\mathrm{rob}, Prop, \{\neg, \wedge, \geq\})$

Plan

wCTL* **is a fragment of** wMSO

Definition: Translation of boolean path formulas

$$\psi ::= \varphi \mid \psi \wedge \psi \mid \neg\psi \mid \psi \,\mathsf{SU}\, \psi$$

Implicitely, ψ has two free variables, the path (set of nodes) and the current node.

We associate with each path formula $\psi \in$ wCTL$^*(\mathbb{K}, \mathit{Prop}, \mathcal{C})$ a boolean MSO formula $\underline{\psi}(x, X) \in$ bMSO$(\mathbb{K}, \Sigma, \mathcal{C})$.

$$\underline{\varphi}(x, X) = (\overline{\varphi}(x) \neq \mathbf{0})$$

$$\underline{\psi_1 \wedge \psi_2}(x, X) = \underline{\psi_1}(x, X) \wedge \underline{\psi_2}(x, X)$$

$$\underline{\neg\psi}(x, X) = \neg\underline{\psi}(x, X)$$

$$\underline{\psi_1 \,\mathsf{SU}\, \psi_2}(x, X) = \underline{\exists} z.(z \in X \wedge x < z \wedge \underline{\psi_2}(z, X) \wedge \forall y.((x < y < z) \xrightarrow{+} \underline{\psi_1}(y, X)))$$

We assume that the interpretation of X is indeed a path.

We use $\underline{\exists}$, $\underline{\vee}$ and $\xrightarrow{+}$ to get boolean formulas.

wCTL* **is a fragment of** wMSO

Definition: Translation of quantitative state formulas

$$\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)}) \mid \mu(\psi)$$

Here, φ only has an implicit free variables, the current node.

We associate with each state formula $\varphi \in$ wCTL$^*(\mathbb{K}, \mathit{Prop}, \mathcal{C})$ a weighted MSO formula $\overline{\varphi}(x) \in$ bMSO$(\mathbb{K}, \Sigma, \mathcal{C})$.

$$[\![\mu(\psi)]\!](t,u) = \bigoplus_{w \in \mathrm{Branches}(t) \mid t,w,u \models \psi} \quad \bigotimes_{v \mid u < v \leq w} \kappa_t(v)$$

$$\overline{\mu(\psi)}(x) = \exists X.(\mathrm{path}(x,X) \wedge \underline{\psi}(x,X) \wedge \xi(x,X))$$

$$\begin{aligned} \mathrm{path}(x,X) = {} & x \in X \\ & \wedge \forall z.(z \in X \xrightarrow{+} (z = x \underline{\vee} \underline{\exists} y.(y \in X \wedge y \lessdot z))) \\ & \wedge \neg \underline{\exists} y, z, z' \in X.(y \lessdot z \wedge y \lessdot z' \wedge z \neq z') \\ & \wedge \forall y.(\ (y \in X \wedge \underline{\exists} z.(y < z)) \xrightarrow{+} \underline{\exists} z.(z \in X \wedge y < z)\ ) \end{aligned}$$

$$\xi(x,X) = \forall y.((y \in X \wedge x < y) \xrightarrow{+} \kappa(y))$$

wCTL* **is a fragment of** wMSO

Theorem:

The translation is correct for finite trees on all semirings.

wCTL **is a fragment of** wMSO **on gPFA**

Unfoldings of probabilistic systems ($\textsc{gPFA}$) are infinite.

The translation of $\overline{\mu(\psi)}(x)$ given above does not work.
We need to be careful with the induced infinite sums and products.

wCTL **is a fragment of** wMSO **on gPFA**

Unfoldings of probabilistic systems (GPFA) are infinite.

The translation of $\overline{\mu(\psi)}(x)$ given above does not work.
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Theorem:
wCTL is a fragment of wMSO on probabilistic systems (GPFA).

Recall the syntax of wCTL:

$$\varphi ::= k \mid \kappa \mid p \mid \bowtie(\varphi_1, \ldots, \varphi_{\mathrm{arity}(\bowtie)}) \mid \mu(\varphi \; \mathsf{SU}^{\leq n} \; \varphi)$$

where $p \in \mathit{Prop}$, $k \in K$, $\bowtie \in \mathcal{C}$, $n \in \mathbb{N} \cup \{\infty\}$.

wCTL **is a fragment of** wMSO **on gPFA**

Definition: Translation of $\mu(\varphi_1 \mathsf{SU}^{\leq n} \varphi_2)$

$$\overline{\mu(\varphi_1 \mathsf{SU}^{\leq n} \varphi_2)}(x) = \exists X.(\mathrm{path}^{\leq n}(x,X) \wedge \underline{\psi}(x,X) \wedge \xi(x,X))$$

$$\mathrm{path}^{\leq\infty}(x,X) = x \in X$$
$$\wedge \forall z.(z \in X \xrightarrow{+} (z = x \mathbin{\underline{\vee}} \underline{\exists}\, y.(y \in X \wedge y \lessdot z)))$$
$$\wedge \neg \underline{\exists}\, y, z, z' \in X.(y \lessdot z \wedge y \lessdot z' \wedge z \neq z')$$

if $n \in \mathbb{N}$, $$\mathrm{path}^{\leq n}(x,X) = \mathrm{path}^{\leq\infty}(x,X) \wedge \neg \underline{\exists}\, x_0 \dots \underline{\exists}\, x_n.$$
$$(x_0 \in X \wedge \cdots \wedge x_n \in X \wedge x < x_0 < x_1 < \cdots < x_n)$$

$$\psi = (\varphi_1 \wedge \neg\varphi_2)\ \mathsf{SU}\ (\varphi_2 \wedge \neg(\mathbf{0}\ \mathsf{SU}\ \mathbf{1}))$$

$$\xi(x,X) = \forall y.((y \in X \wedge x < y) \xrightarrow{+} \kappa(y))$$

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$$\mathrm{path}^{\leq \infty}(x,X) = x \in X$$
$$\wedge \forall z.(z \in X \xrightarrow{+} (z = x \,\underline{\vee}\, \underline{\exists} y.(y \in X \wedge y \lessdot z)))$$
$$\wedge \neg \underline{\exists} y, z, z' \in X.(y \lessdot z \wedge y \lessdot z' \wedge z \neq z')$$

if $n \in \mathbb{N}$, $\quad \mathrm{path}^{\leq n}(x,X) = \mathrm{path}^{\leq \infty}(x,X) \wedge \neg \underline{\exists} x_0 \ldots \underline{\exists} x_n.$
$$(x_0 \in X \wedge \cdots \wedge x_n \in X \wedge x < x_0 < x_1 < \cdots < x_n)$$

$$\psi = (\varphi_1 \wedge \neg\varphi_2) \,\mathsf{SU}\, (\varphi_2 \wedge \neg(\mathbf{0} \,\mathsf{SU}\, \mathbf{1}))$$

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$\mathrm{path}^{\leq n}(x,X) \wedge \underline{\psi}(x,X)$ is a boolean formula which holds if and only if X is a minimal path satisfying $\varphi_1 \,\mathsf{SU}^{\leq n}\, \varphi_2$.

wCTL is a fragment of wMSO on gPFA

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$\exists X$ computes the sum of the probability of such paths.

wCTL **is a fragment of** wMSO **on gPFA**

Theorem:

The translation is correct for infinite trees which are unfoldings of GPFA.

Corollary:

PCTL is a fragment of wMSO on GPFA

Plan

Conclusion

- There is a very rich theory for probabilistic systems.
  - Various logics for specification
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- Analysis of other quantitative properties is more and more important.
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- We should develop a strong theory for analysis of various quantitative aspects
  Building upon existing theory of weighted automata
  and the large experience in analysing probabilistic systems.

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- Identify fragments for which satisfiability and model checking are decidable.

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- Find a weighted μ-calculus which contains wCTL and compare its expressivity with wMSO.
  Weighted μ-calculus on words Meinecke, DLT'09
  Weighted μ-calculus for quantitative games Fischer, Grädel & Kaiser '08

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  But still not quantitative.
- We need to study bisimulation distances expressing how close two states are.
  See Fahrenberg, Larsen & Thrane '09