# Propositional Dynamic Logic and Asynchronous Cascade Decompositions for Regular Trace Languages

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# Outline

- Model of Concurrency: Mazurkiewicz Traces
- A propositional dynamic logic for traces
- Asynchronous automata (Zielonka) and asynchronous labeling functions
- Cascade product / decomposition (Krohn-Rhodes)
- Conclusion

# Mazurkiewicz Traces

$\mathscr{P} = \{1,2,3\}$

$\Sigma = \{a, b, c, d, e\}$

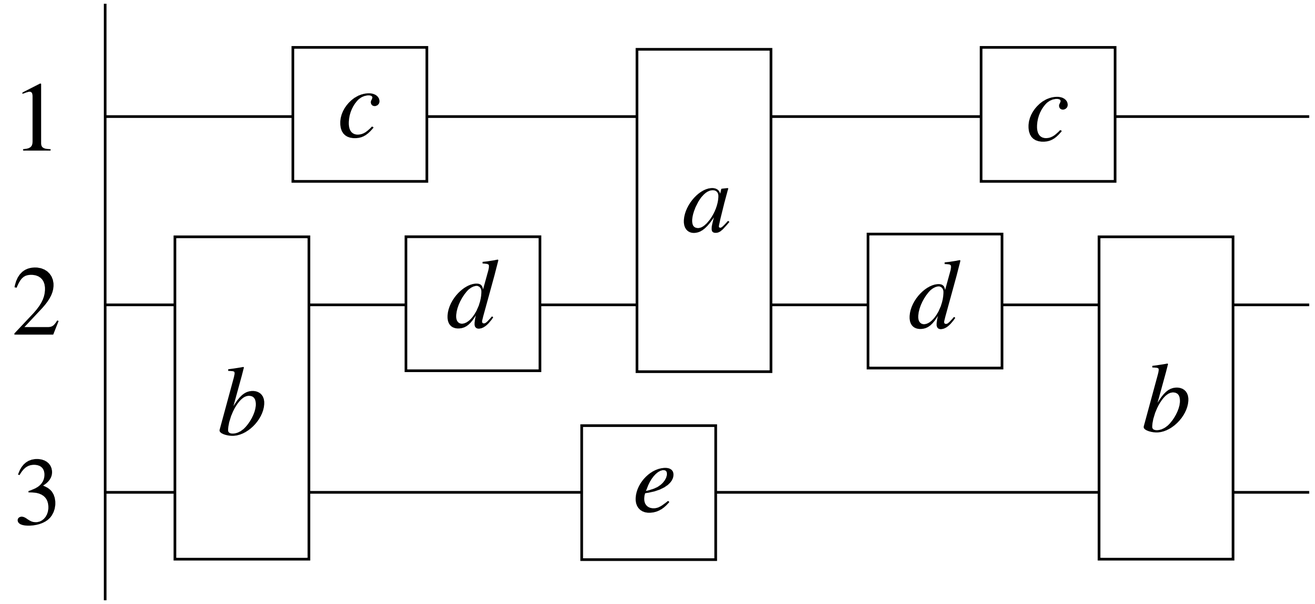
$\mathrm{loc}(a) = \{1,2\}$

$\mathrm{loc}(b) = \{2,3\}$

$\mathrm{loc}(c) = \{1\}$

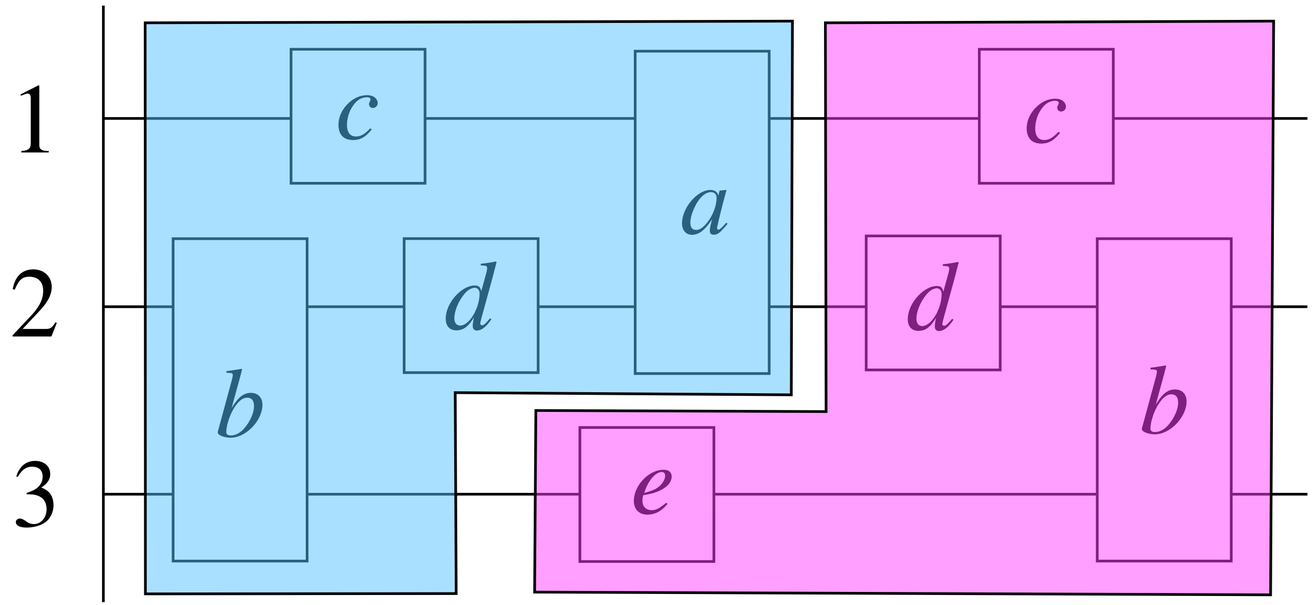
$\mathrm{loc}(d) = \{2\}$

$\mathrm{loc}(e) = \{3\}$




- *set of traces denoted* $Tr(\Sigma, \mathscr{P}, \mathrm{loc})$ *or simply* $Tr(\Sigma)$
- *Trace Language:* $L \subseteq Tr(\Sigma)$

# Concatenation, Independence, Commutation, Monoid



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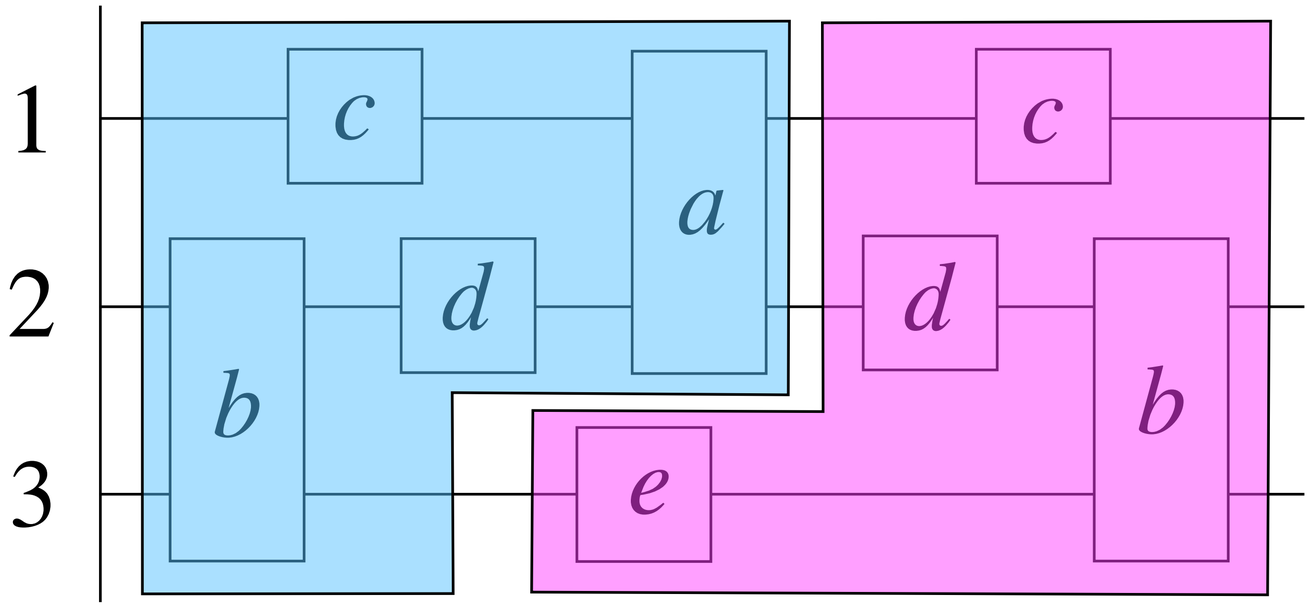


- $Tr(\Sigma)$ *with trace concatenation is a monoid*

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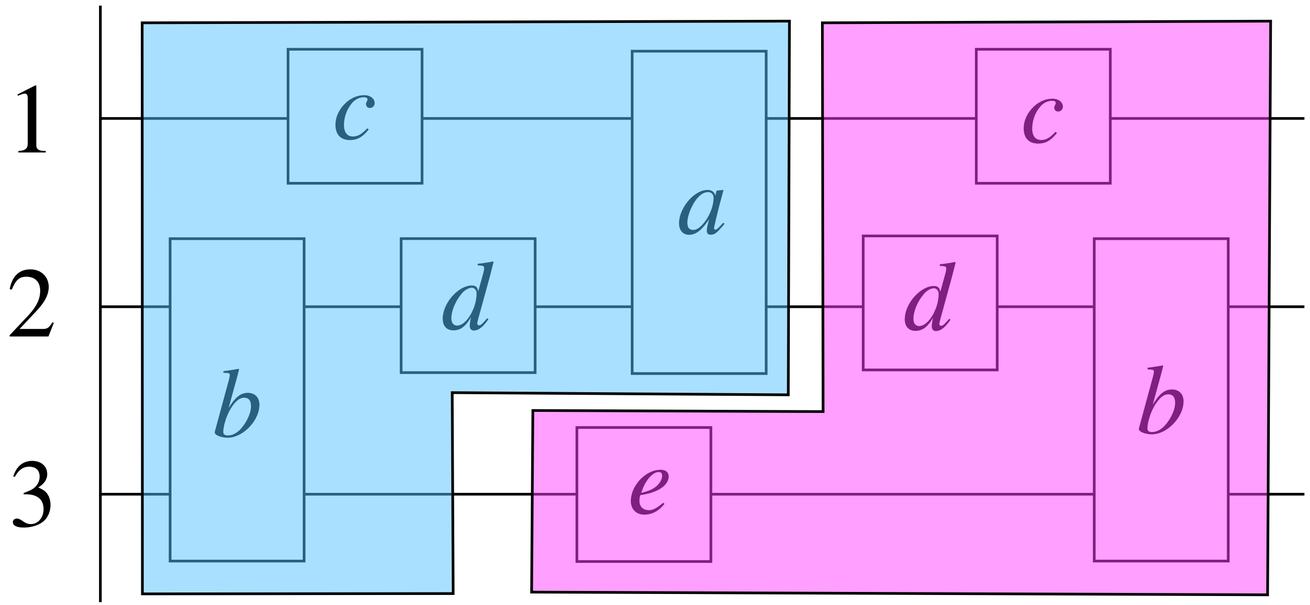


- $x \; I \; y$ iff $\mathrm{loc}(x) \cap \mathrm{loc}(y) = \varnothing$
- $a \; I \; e$
- $a \cdot e = e \cdot a$

- *$Tr(\Sigma)$ with trace concatenation is a monoid*
- *Free partially commutative monoid*

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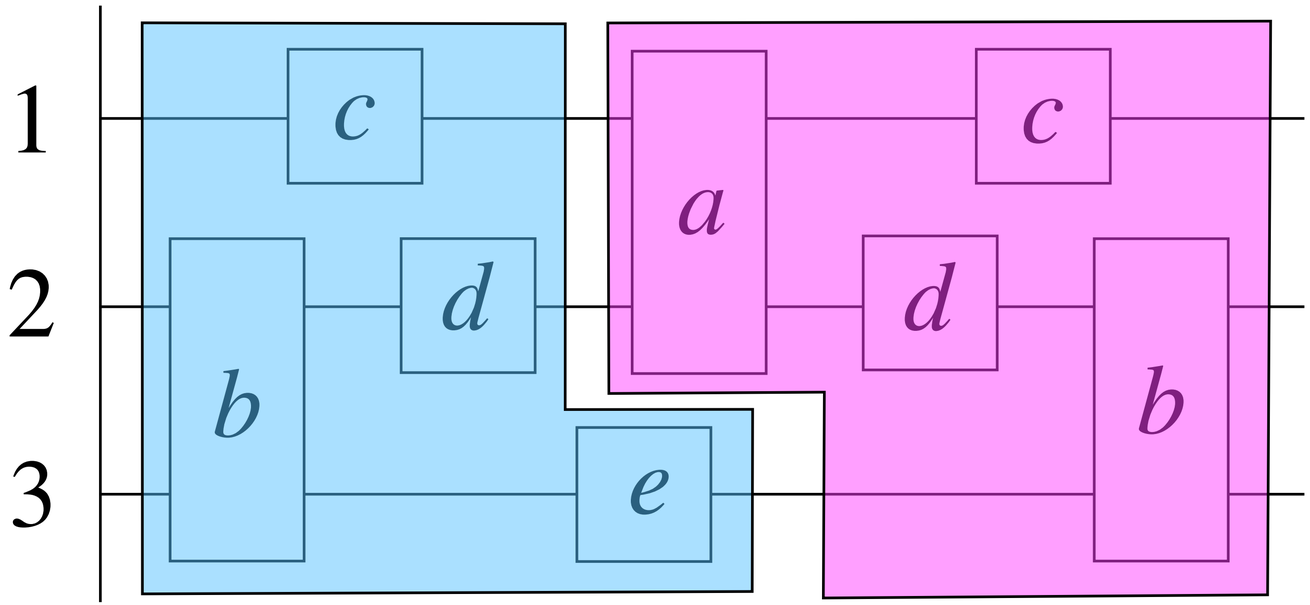


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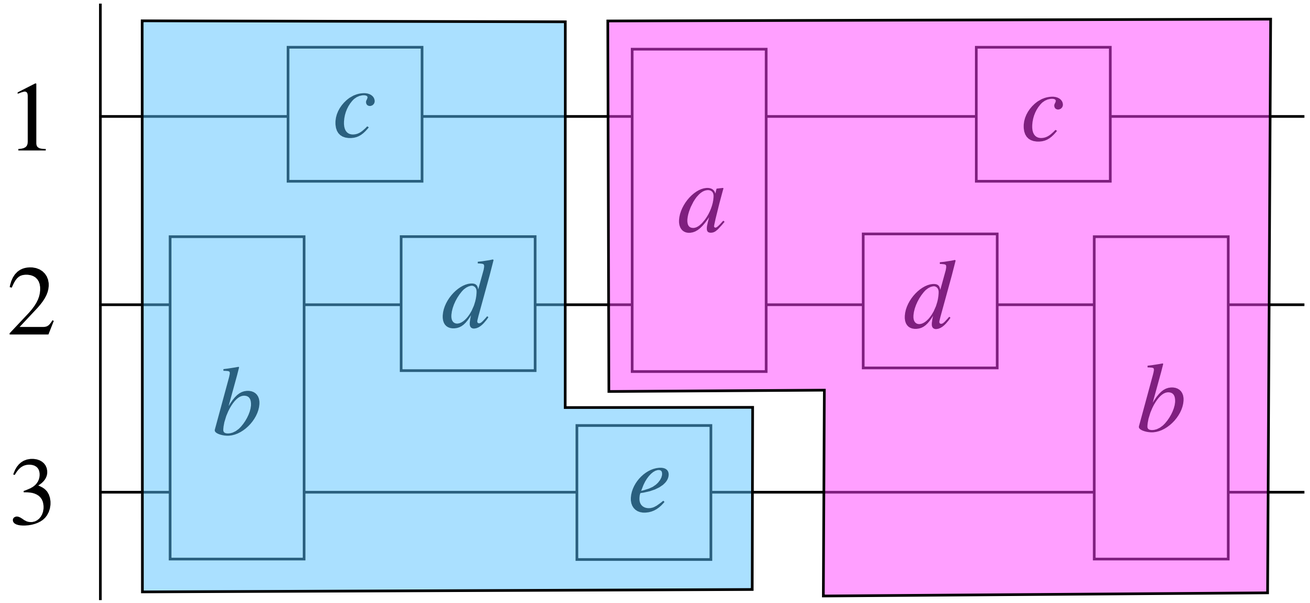


- $x \; I \; y$ iff $\mathrm{loc}(x) \cap \mathrm{loc}(y) = \varnothing$
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- $Tr(\Sigma)$ *with trace concatenation is a monoid*
- *Free partially commutative monoid*
- $L \subseteq Tr(\Sigma)$ *is regular if* $L = \eta^{-1}(\eta(L))$ *for some morphism* $\eta : Tr(\Sigma) \to M$ *(finite monoid)*

| | Expressions | Computation | Logic | Algebra |
|---|---|---|---|---|
| **Regular Trace Languages** | *Regular expressions with restricted star operation* | *Diamond automata* | *Monadic second order logic* | *Morphisms to finite Monoids* |

*Ochmański Muscholl Diekert Gastin Restivo Salemi Thomas et. al.*

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| ∪∤ | | | | |
| **Aperiodic Trace Languages** | *Star-free expressions* | *Counter-free diamond automata* | *First order logic Temporal logic* | *Aperiodic Monoids* |

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  - State formulas $\varphi ::= p \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\varphi$
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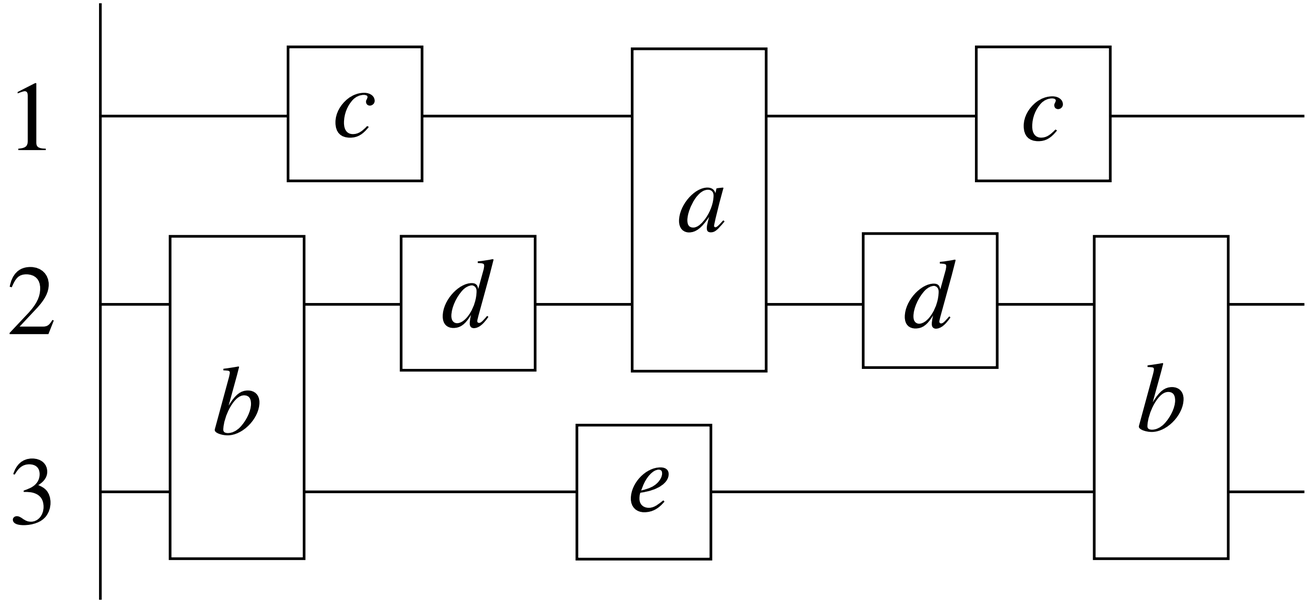
- Linear Dynamic Logic (Giacomo, Vardi 2013) over words

  Regular word languages = MSO definable = LDL definable

# Past PDL for Traces



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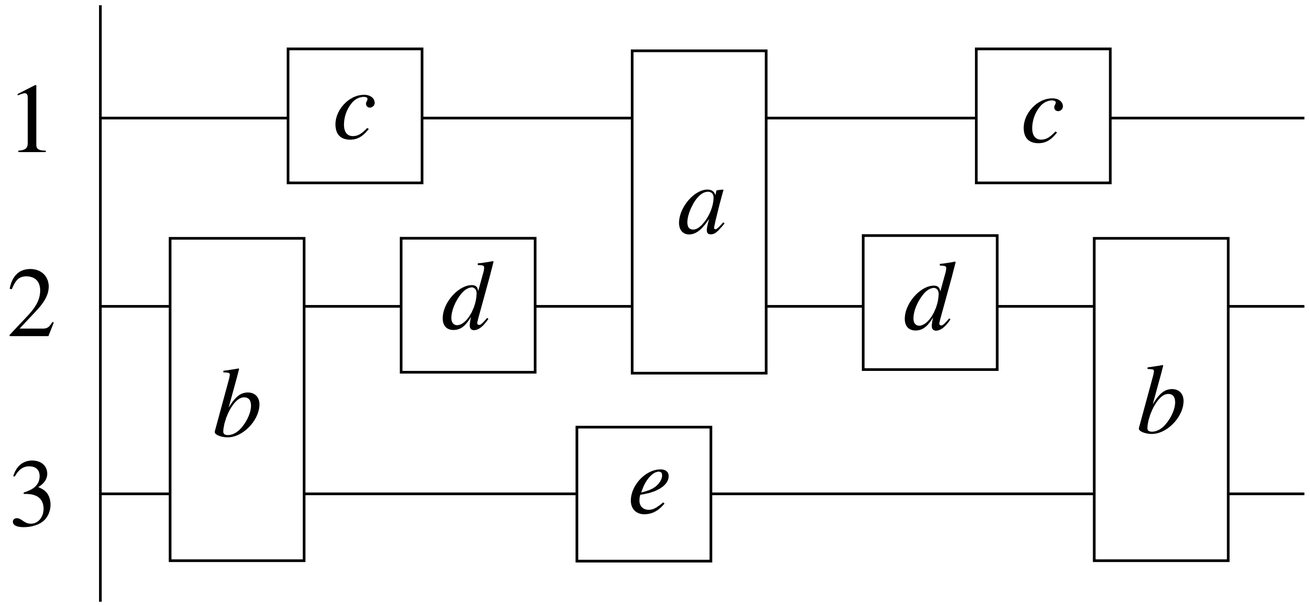


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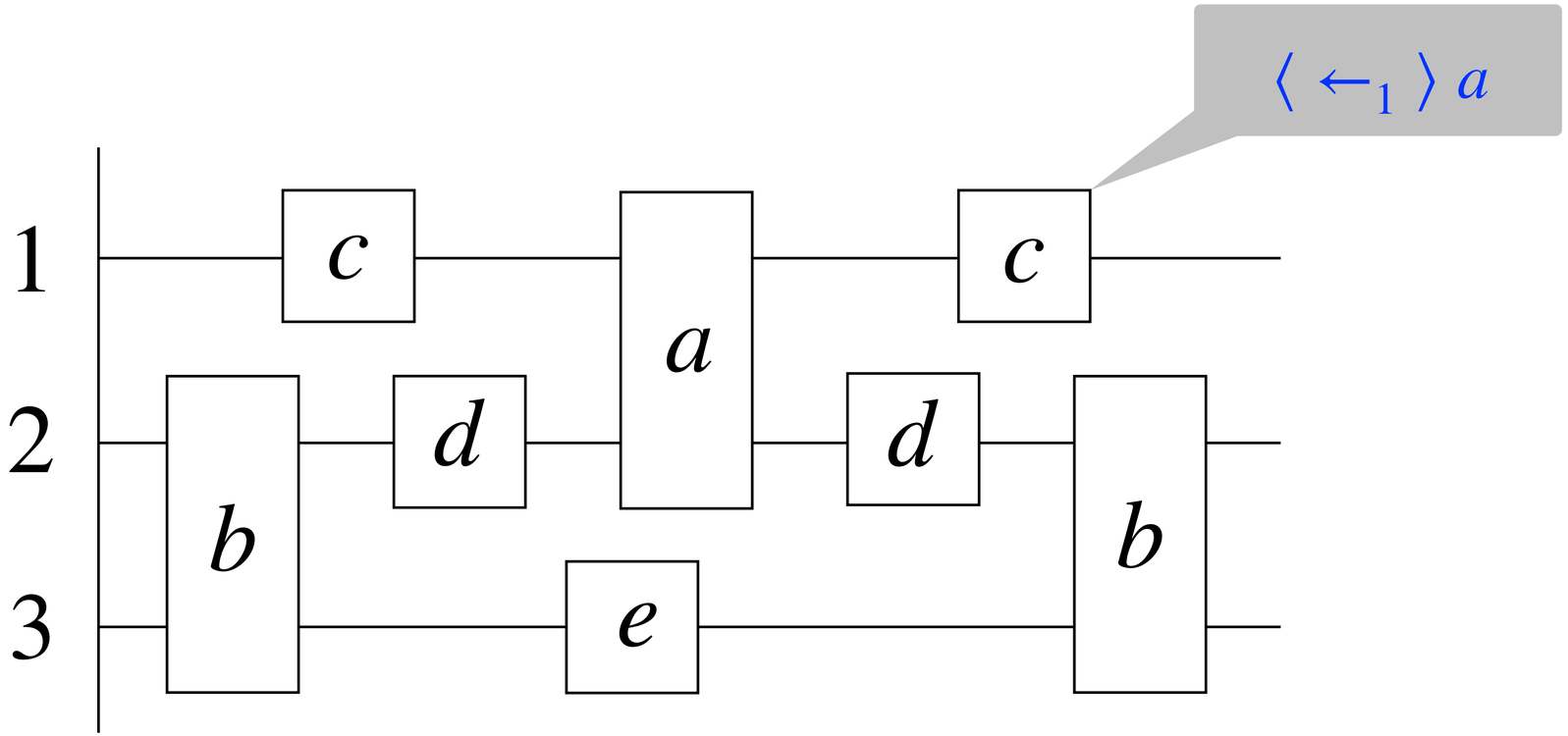


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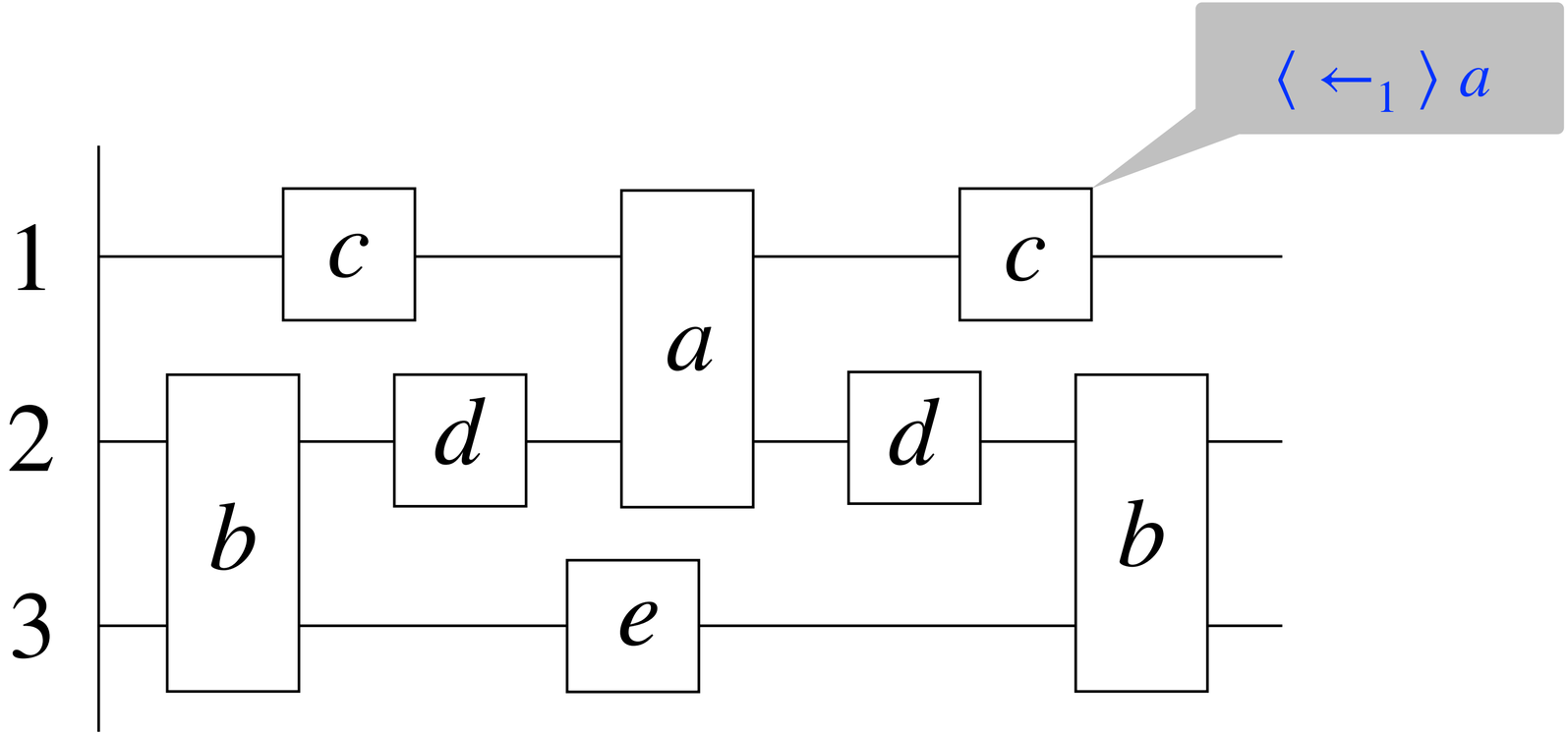


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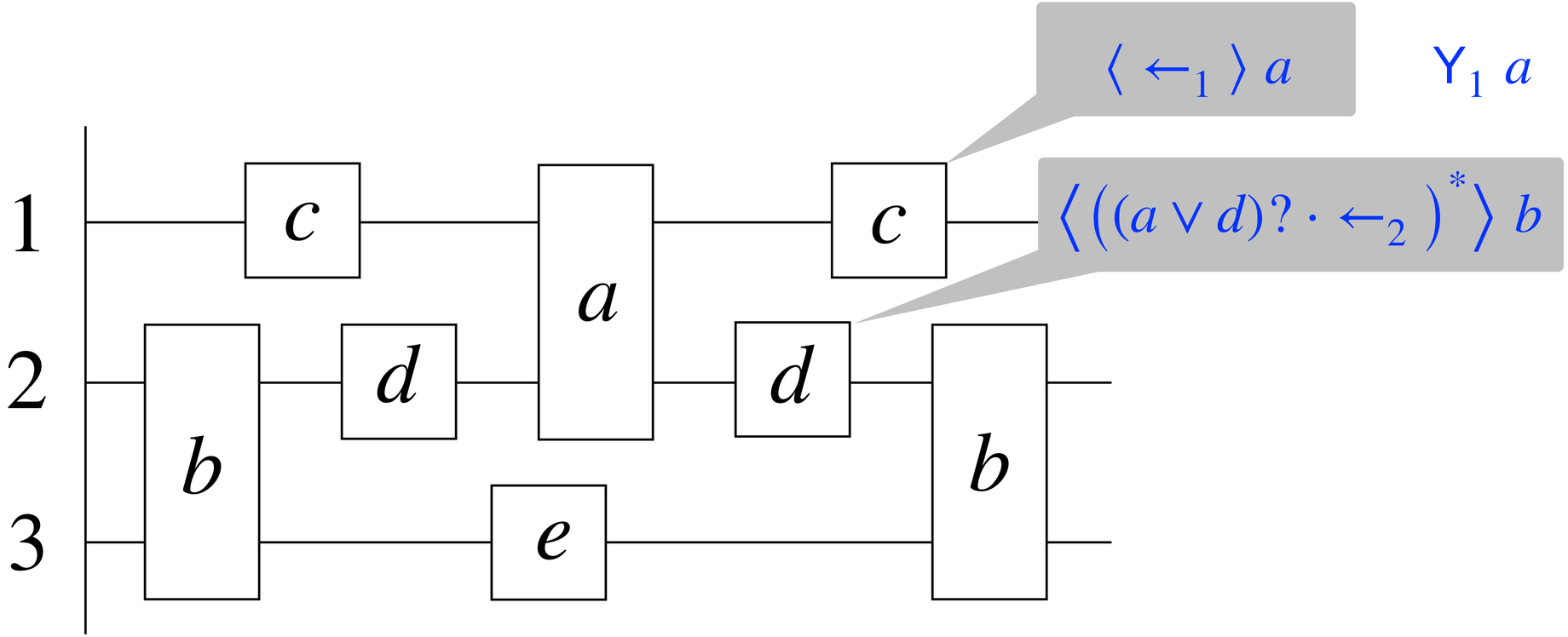


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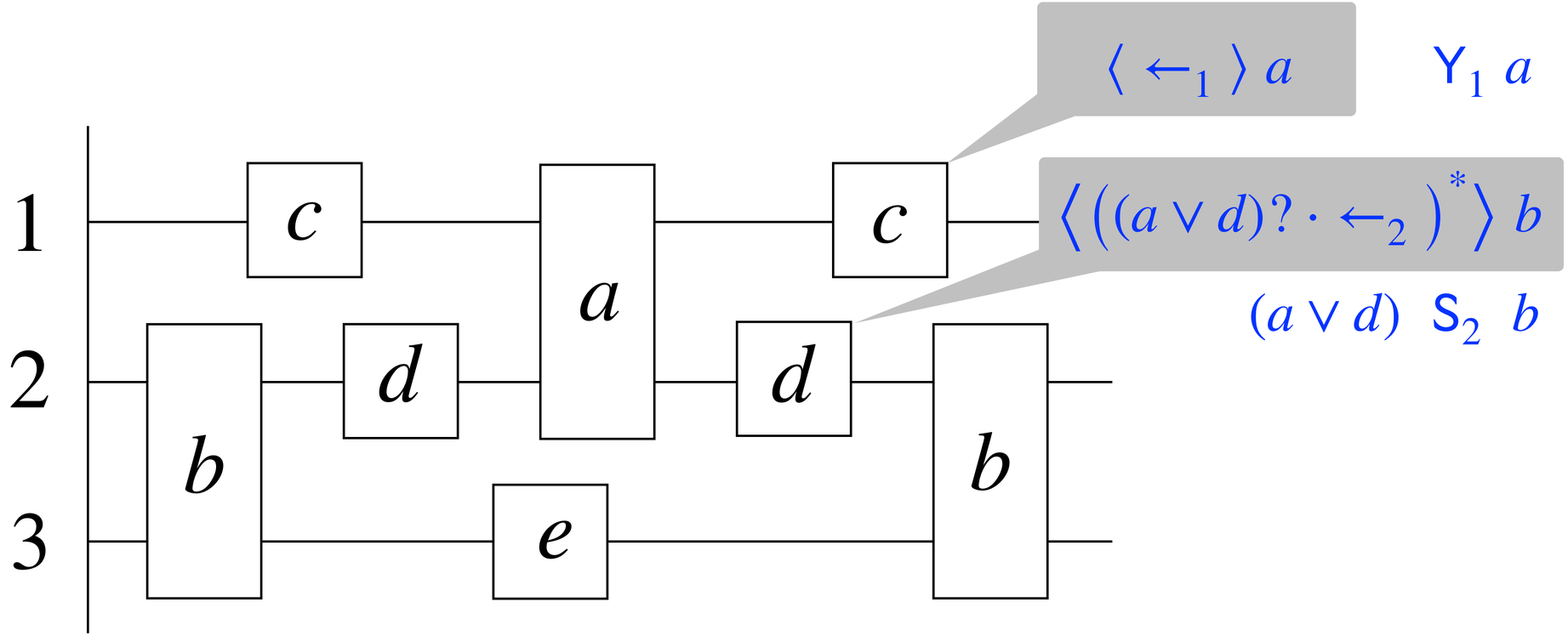


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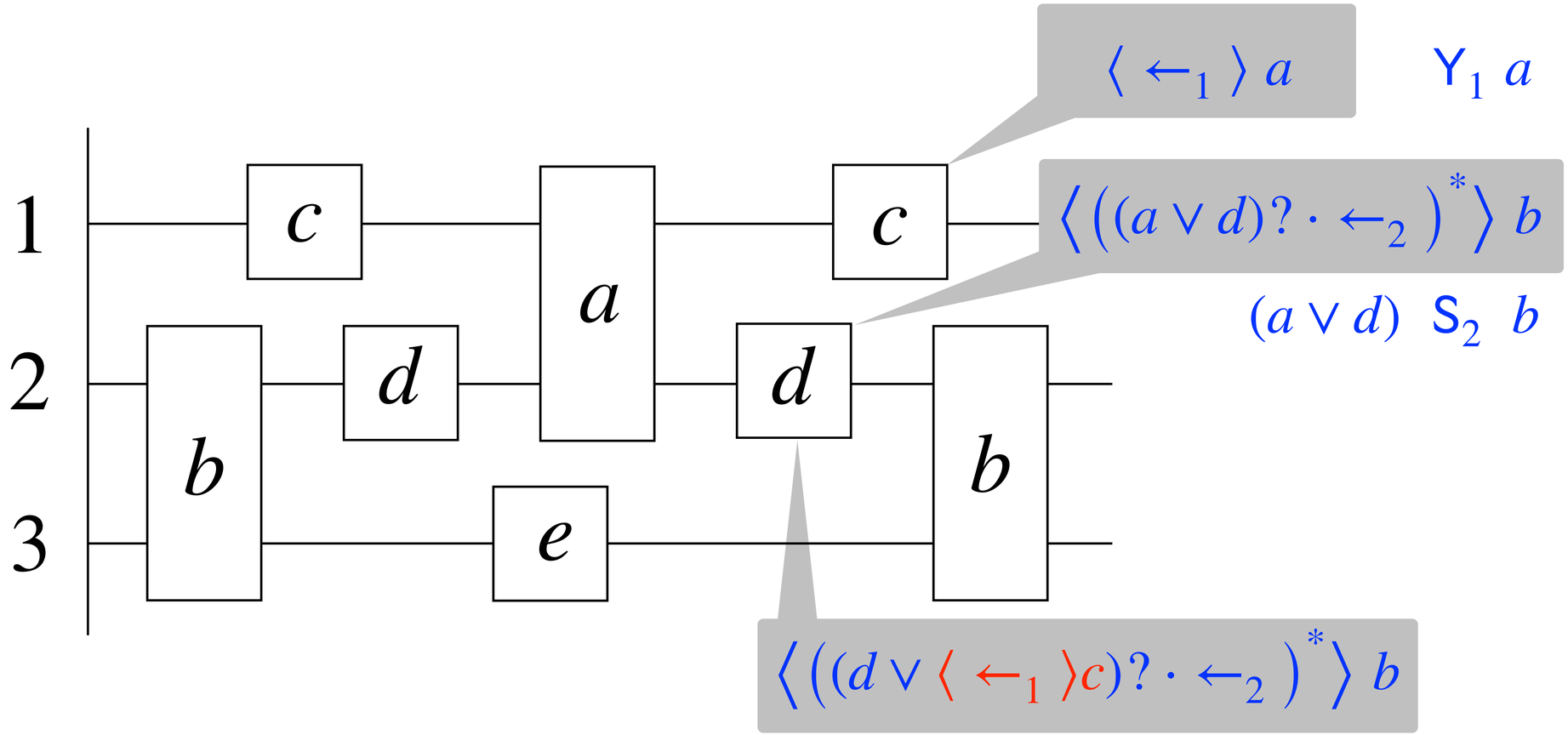


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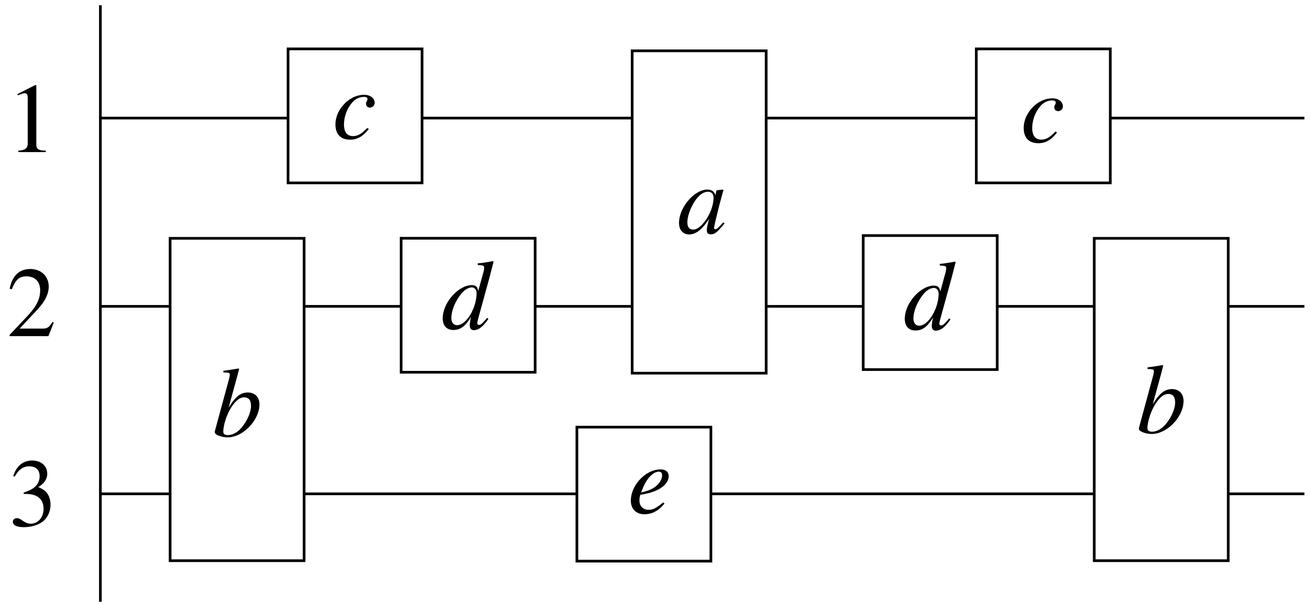


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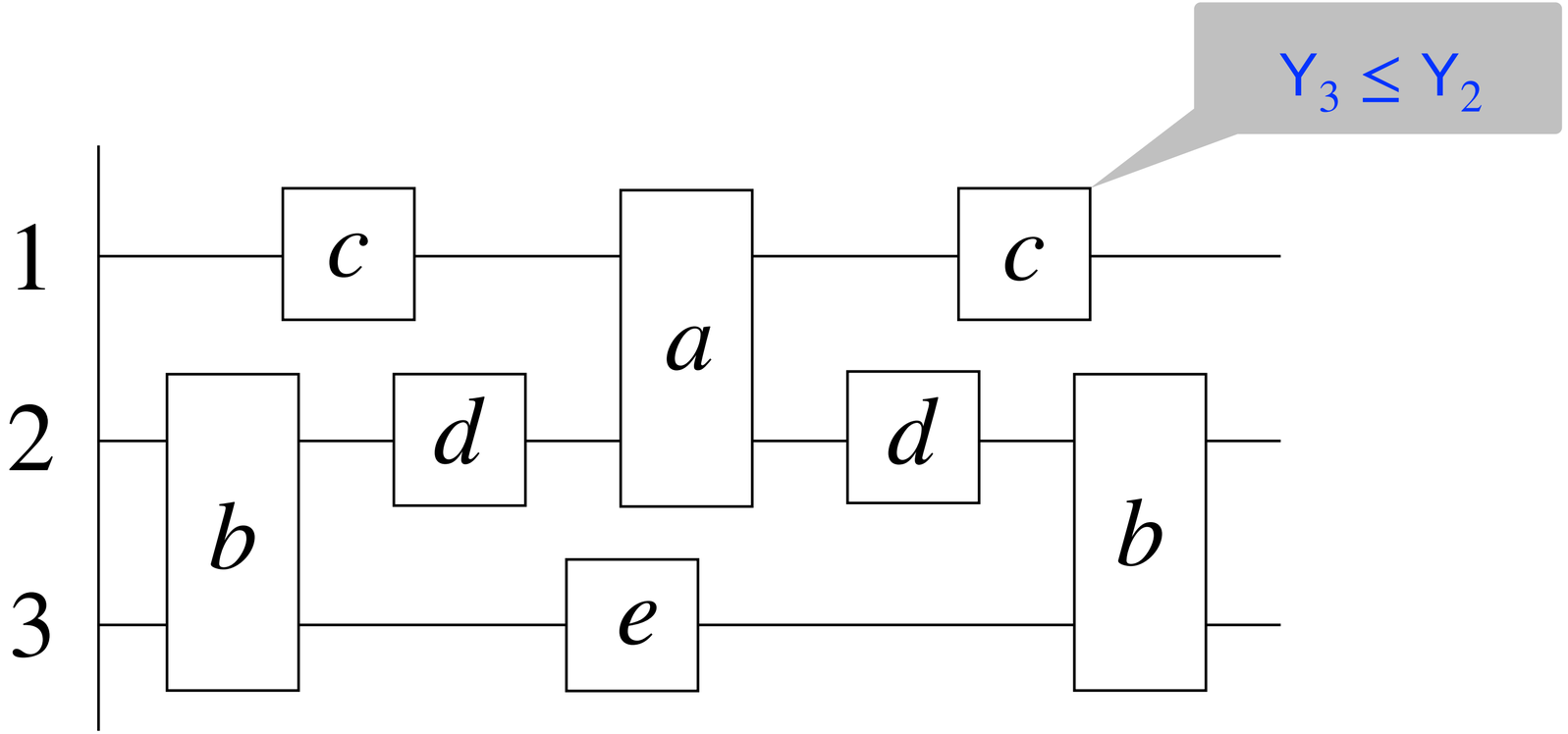


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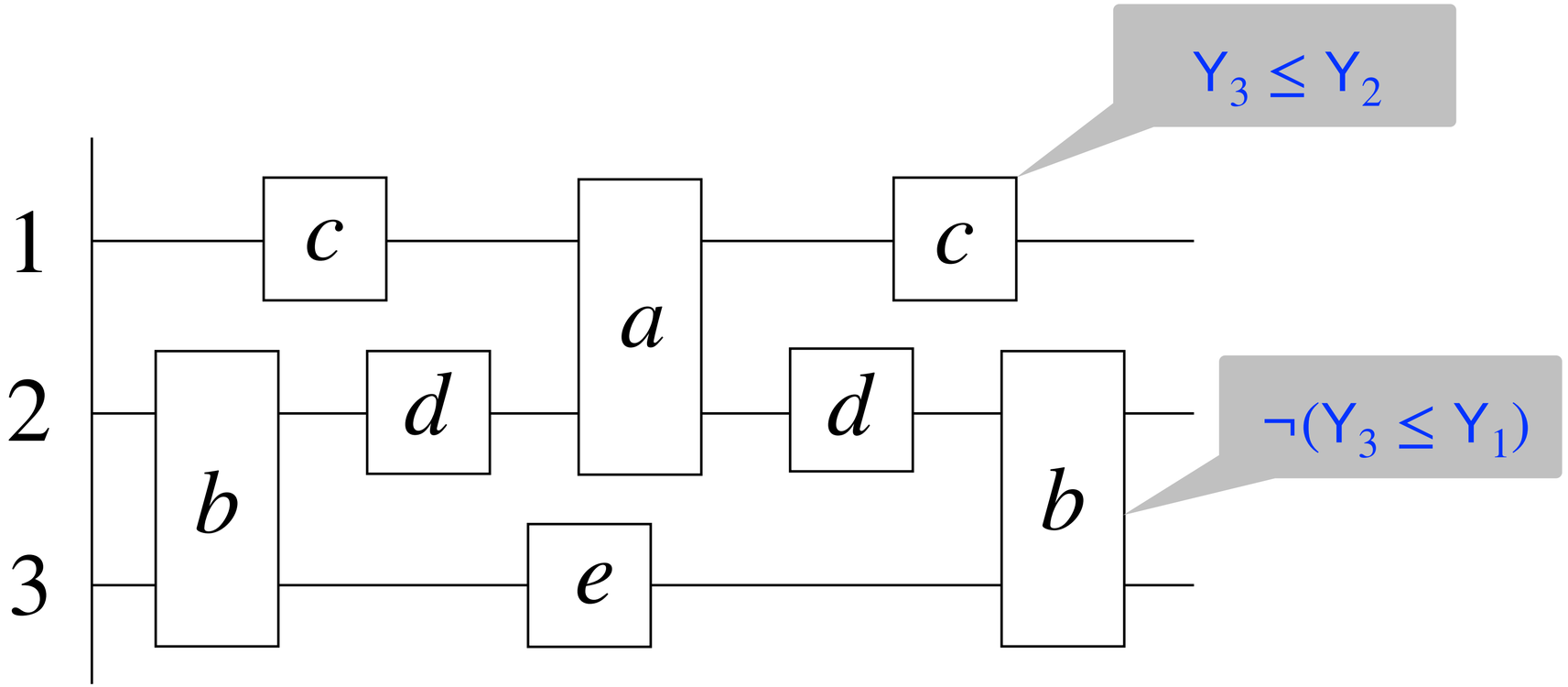


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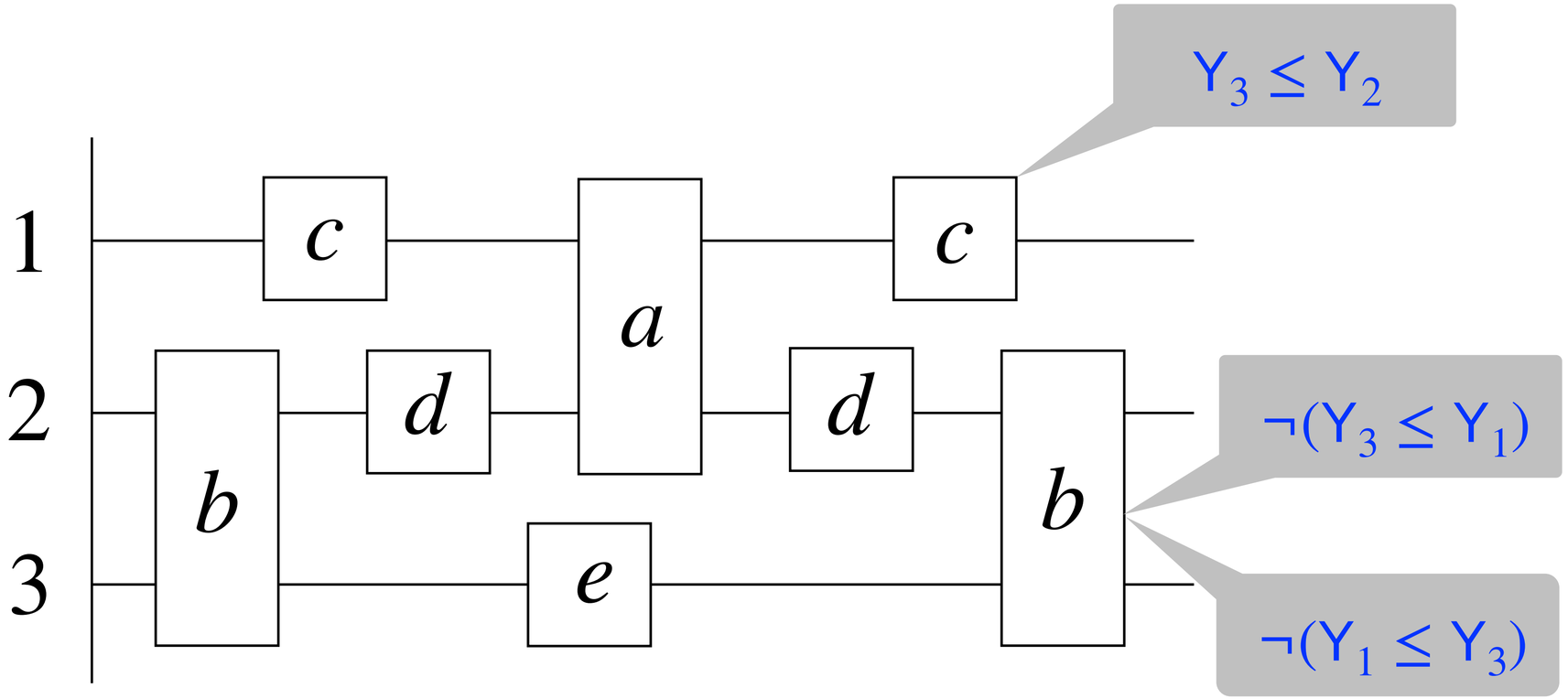


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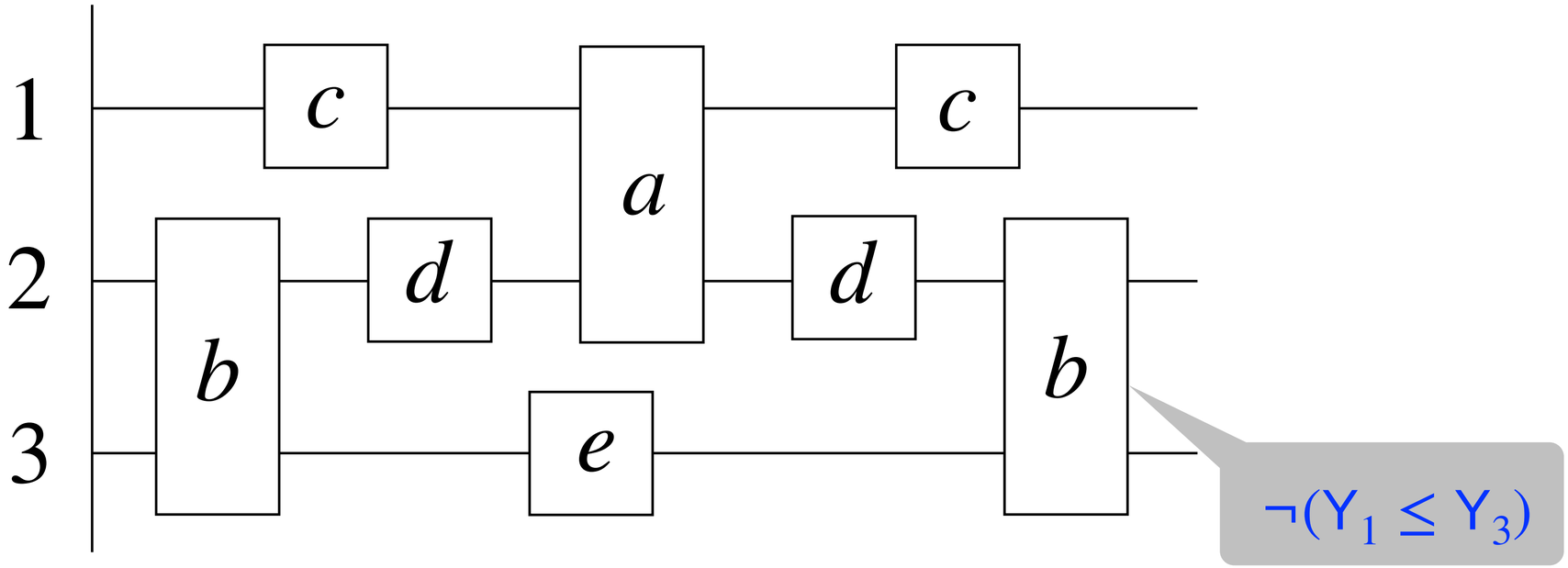


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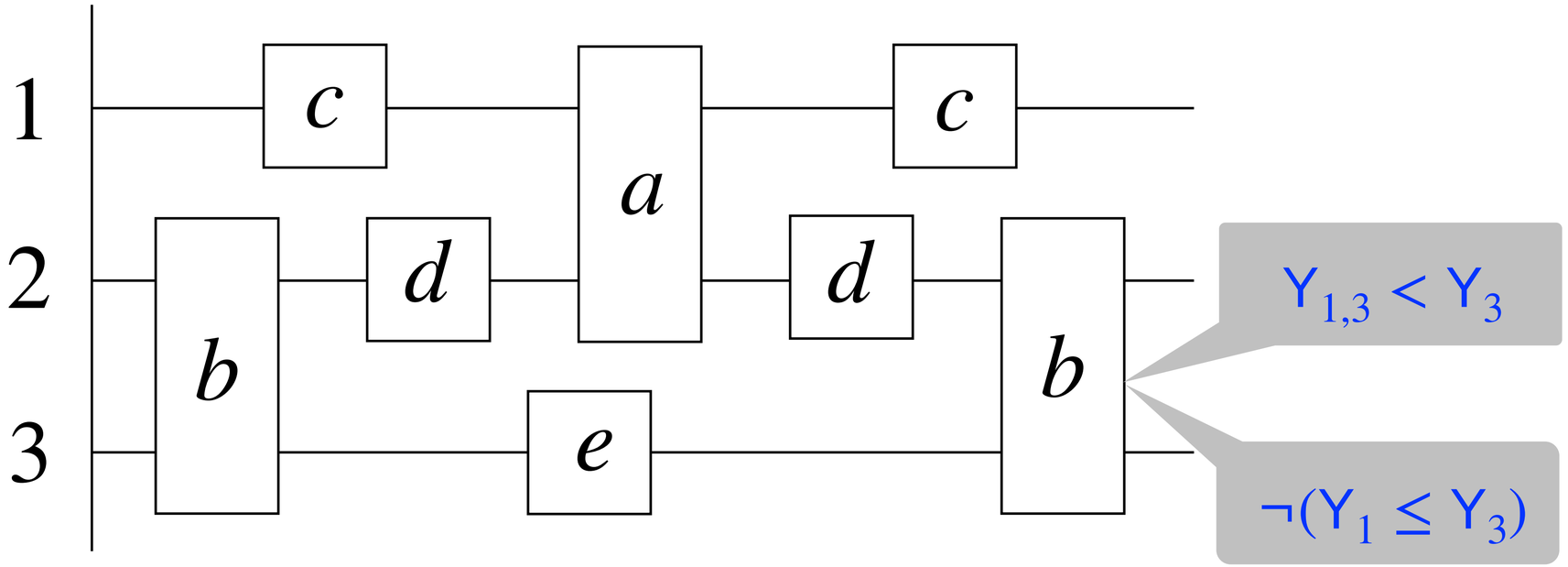


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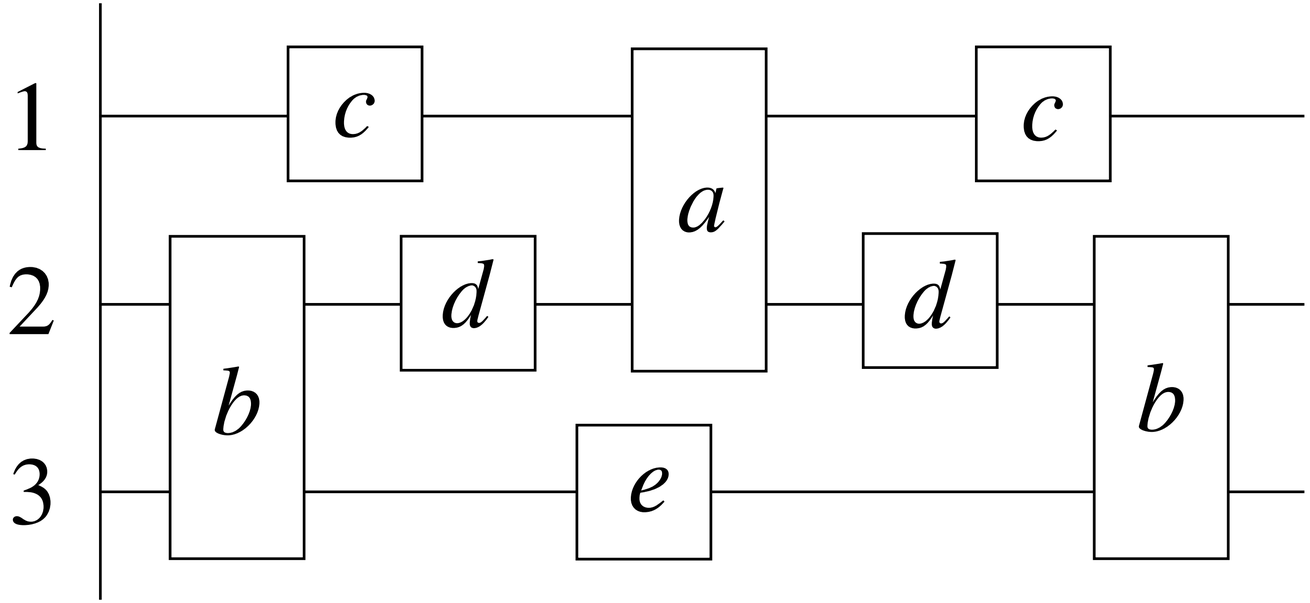


- Trace formulas / Sentences $\Phi ::= \mathsf{EM}_i\ \varphi \mid \Phi \vee \Phi \mid \neg\Phi \mid \mathsf{L}_i \leq \mathsf{L}_j \mid \mathsf{L}_{i,k} \leq \mathsf{L}_j$
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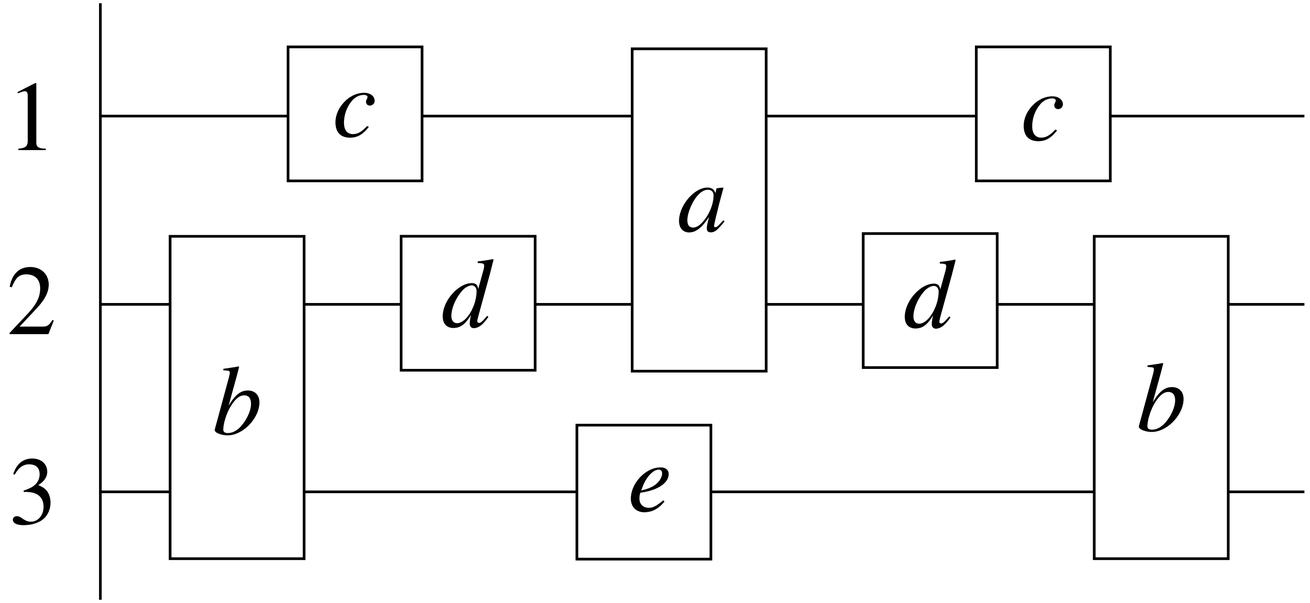


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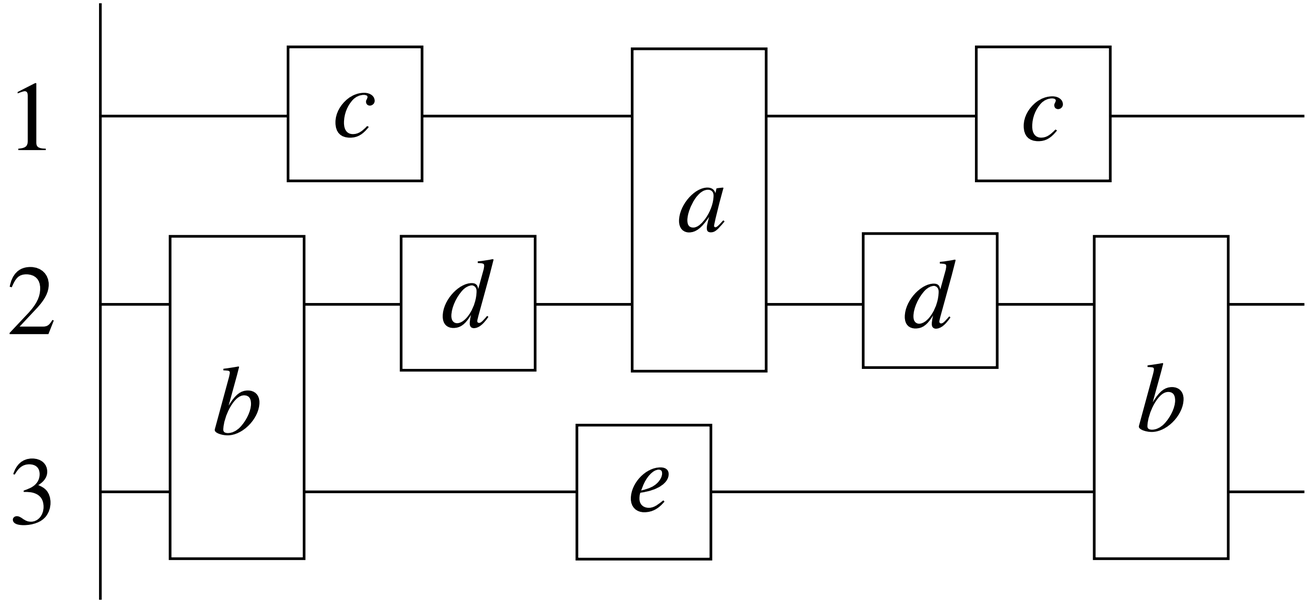


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- $T \models \mathsf{EM}_1\ c$
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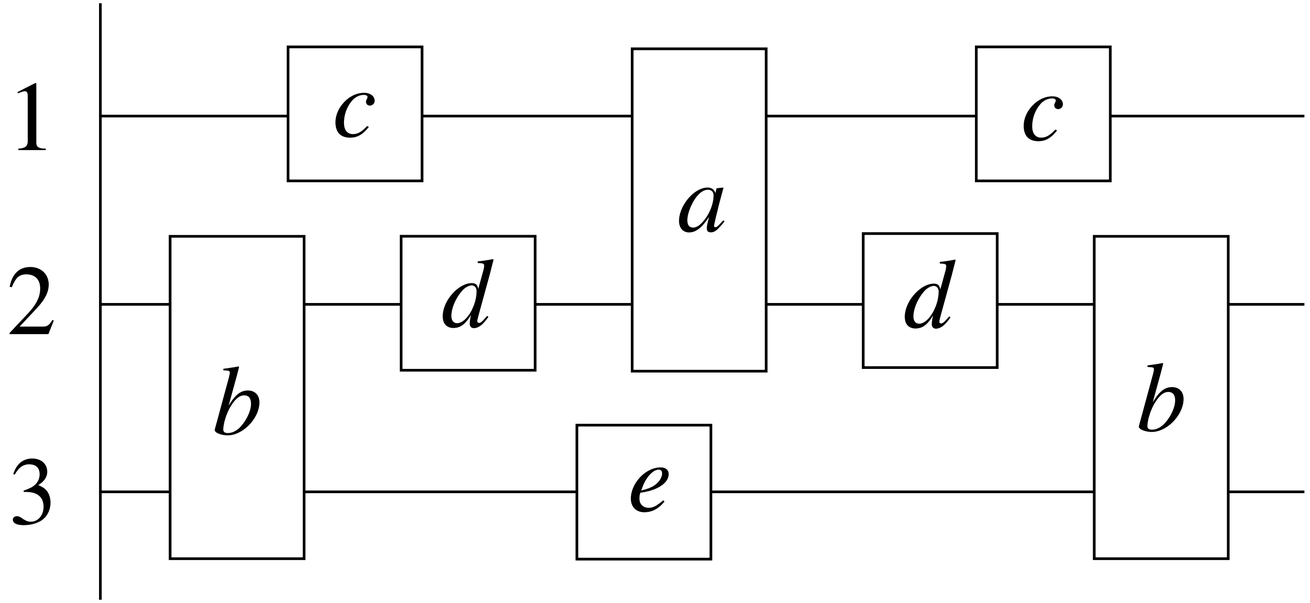
- $T \models \neg(\mathsf{L}_1 \leq \mathsf{L}_3)$

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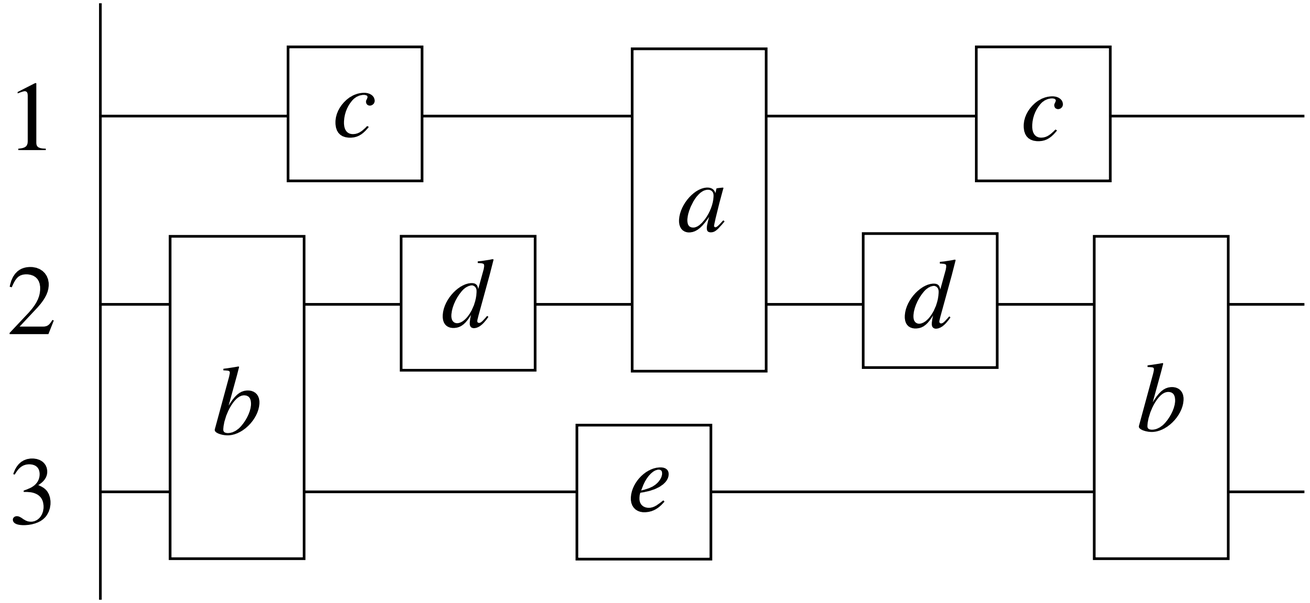
- $T \models \mathsf{L}_{1,2} < \mathsf{L}_3$

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- First, construct a locPastPDL event formula $\varphi^{(m)}$ such that, if $T$ is a prime trace (i.e., having a single maximal event), $\eta(T) = m$ if and only if $T, \max(T) \vDash \varphi^{(m)}$

  (induction on the number of processes, depends crucially on $\mathsf{Y}_i \leq \mathsf{Y}_j$, $\mathsf{Y}_{i,k} \leq \mathsf{Y}_j$)

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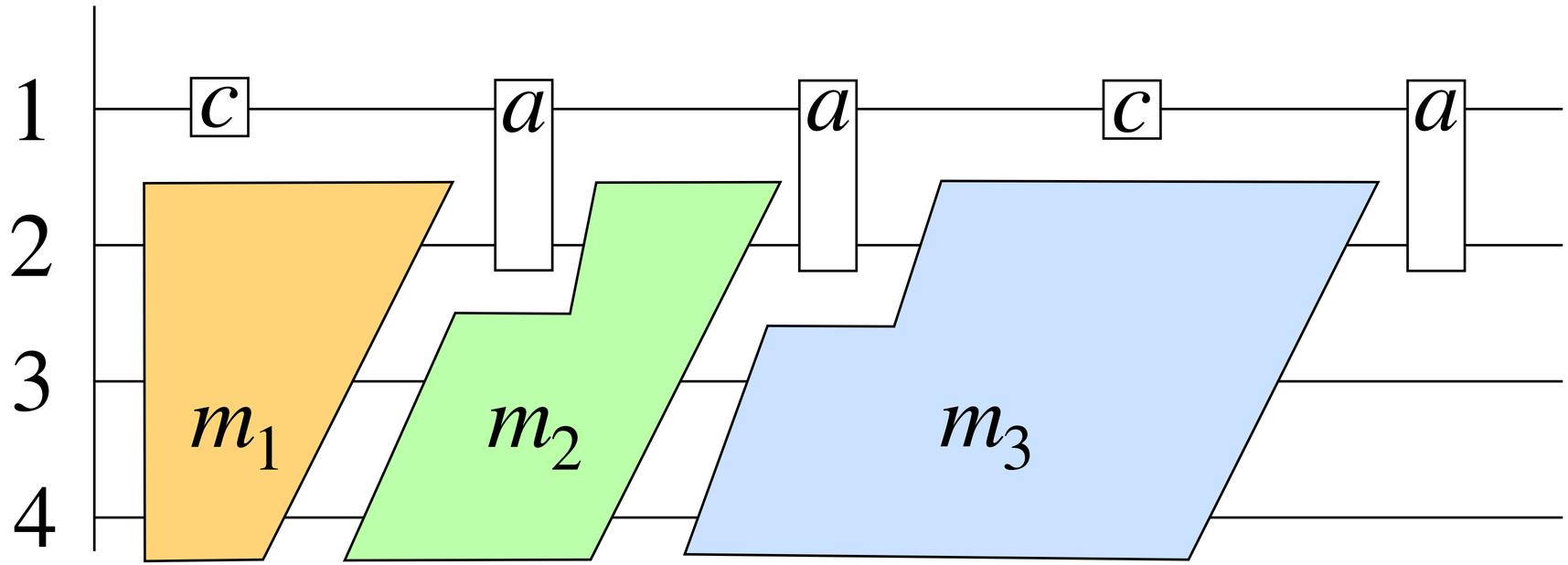

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- First, construct a locPastPDL event formula $\varphi^{(m)}$ such that, if $T$ is a prime trace (i.e., having a single maximal event), $\eta(T) = m$ if and only if $T, \max(T) \models \varphi^{(m)}$

  (induction on the number of processes, depends crucially on $\mathsf{Y}_i \leq \mathsf{Y}_j$, $\mathsf{Y}_{i,k} \leq \mathsf{Y}_j$)
- For the general case, decompose an arbitrary (non prime) trace into a product of prime traces.

# (Local) Past-PDL for Traces

Theorem 1

PastPDL = Regular Trace Languages

locPastPDL

∩ fragment

PastPDL

∩ easy

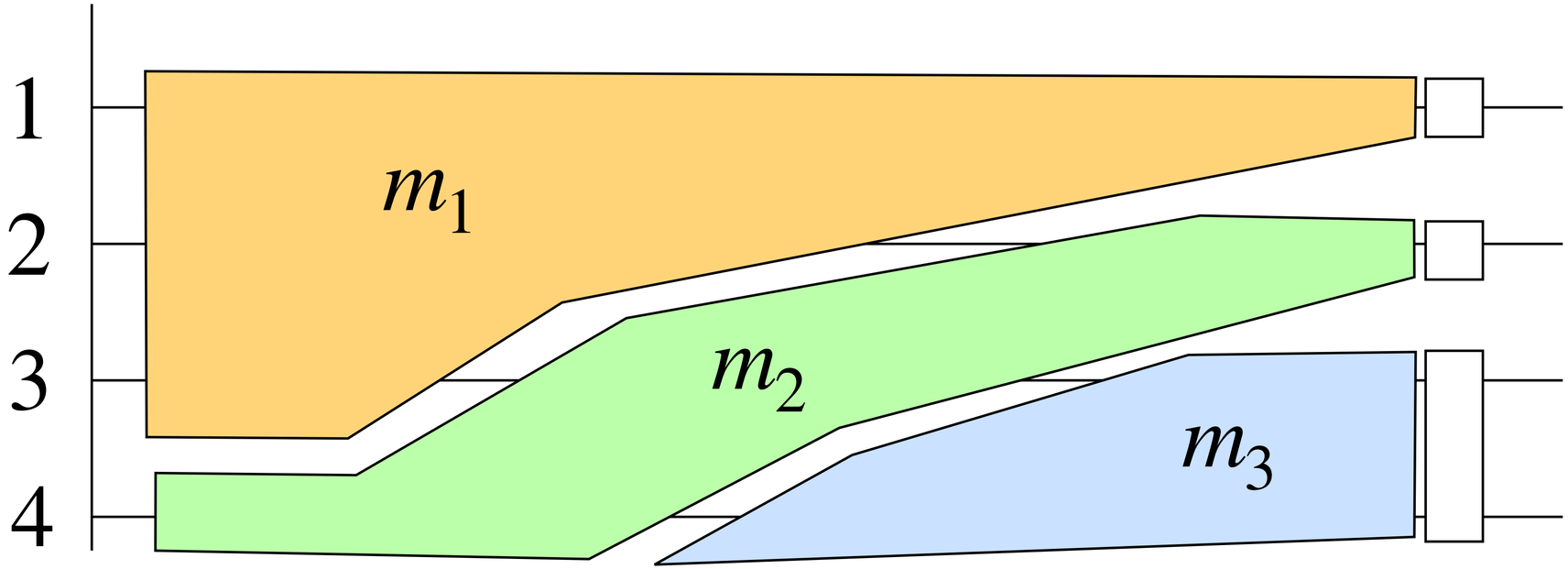
MSO

∩ known

Regular

∩ complex & subtle

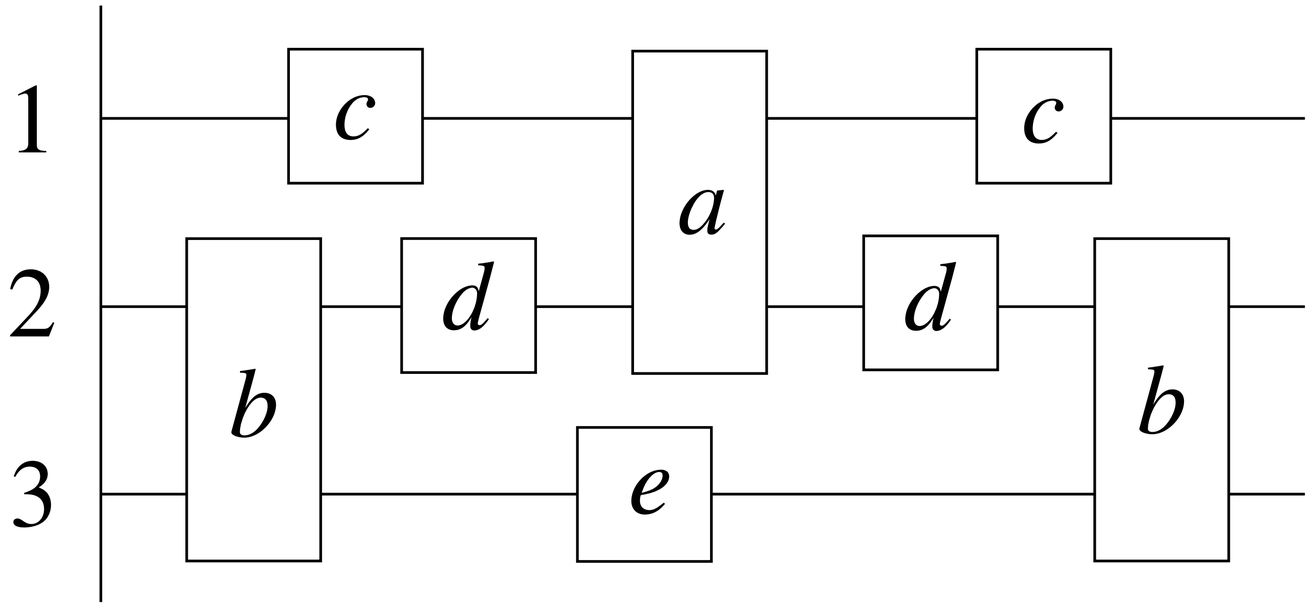
locPastPDL



# Outline

- ✅ Model of Concurrency: Mazurkiewicz Traces
- ✅ A propositional dynamic logic for traces
- Asynchronous automata (Zielonka) and asynchronous labeling functions
- Cascade product / decomposition (Krohn-Rhodes)
- Conclusion

# Asynchronous automata (Zielonka)




$$\mathcal{A} = \left(\{S_i\}_{i\in\mathcal{P}}, \{\delta_a\}_{a\in\Sigma}, s^{in}, F\right)$$

- $S_i$ local states for process $i$
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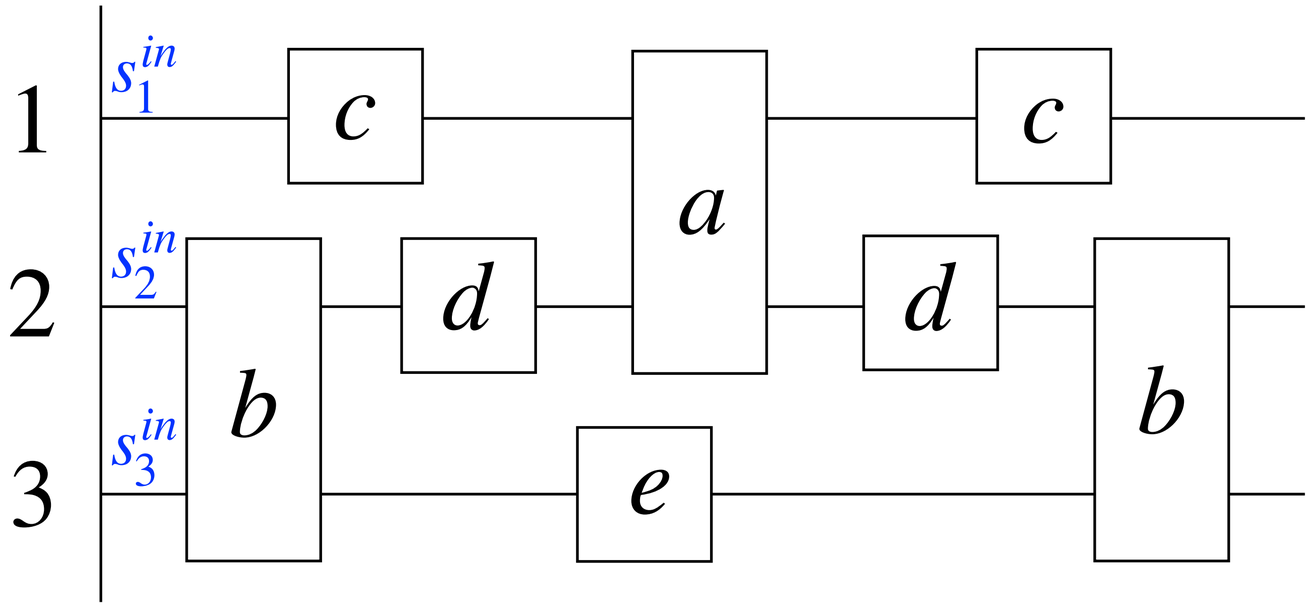


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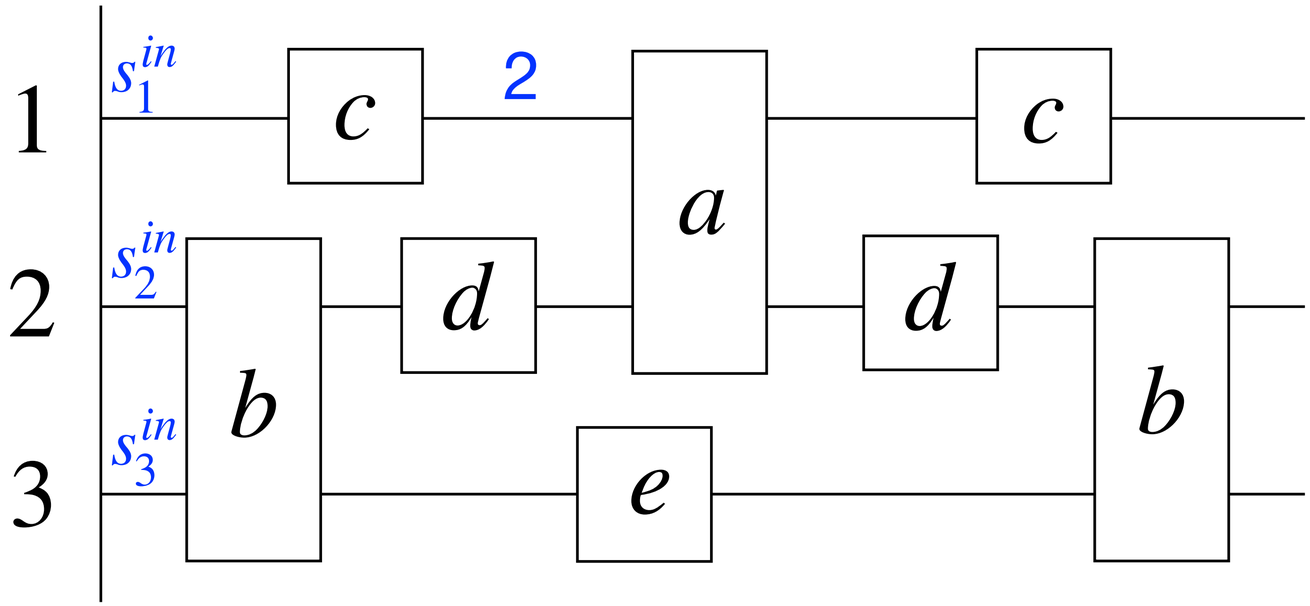


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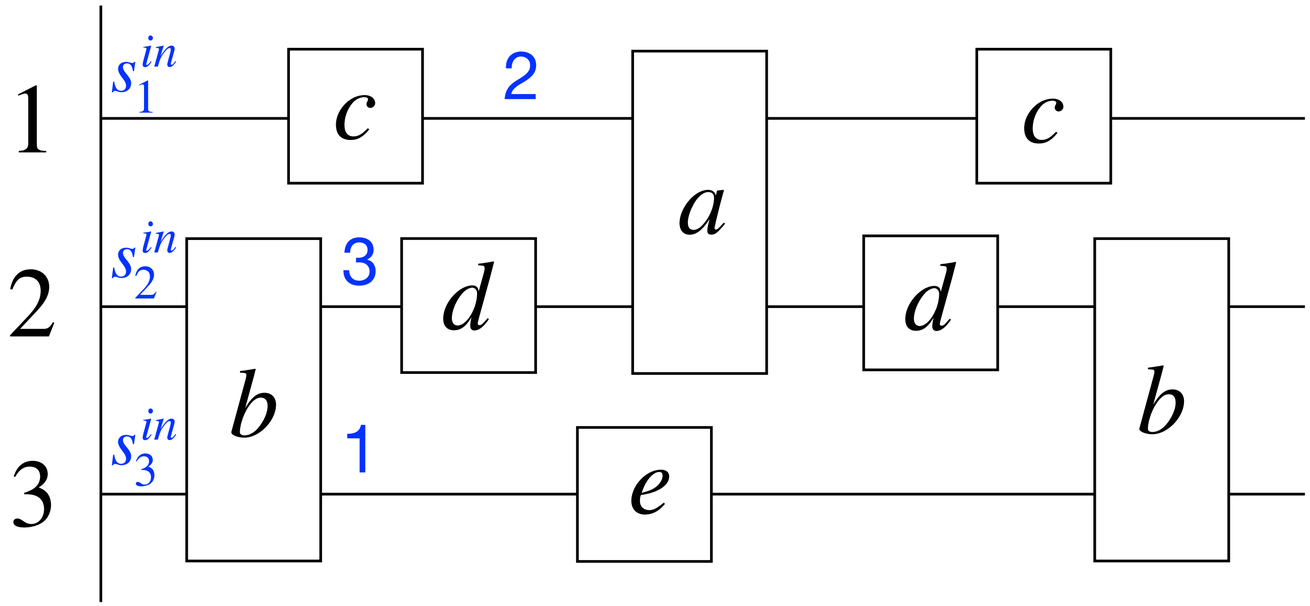


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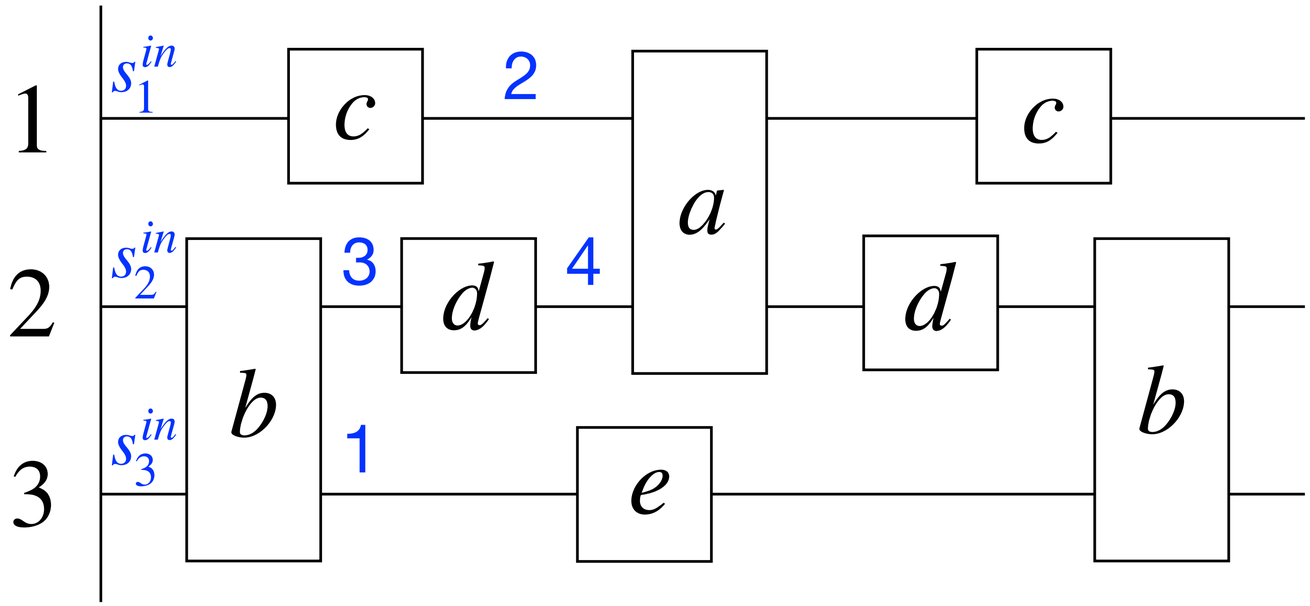


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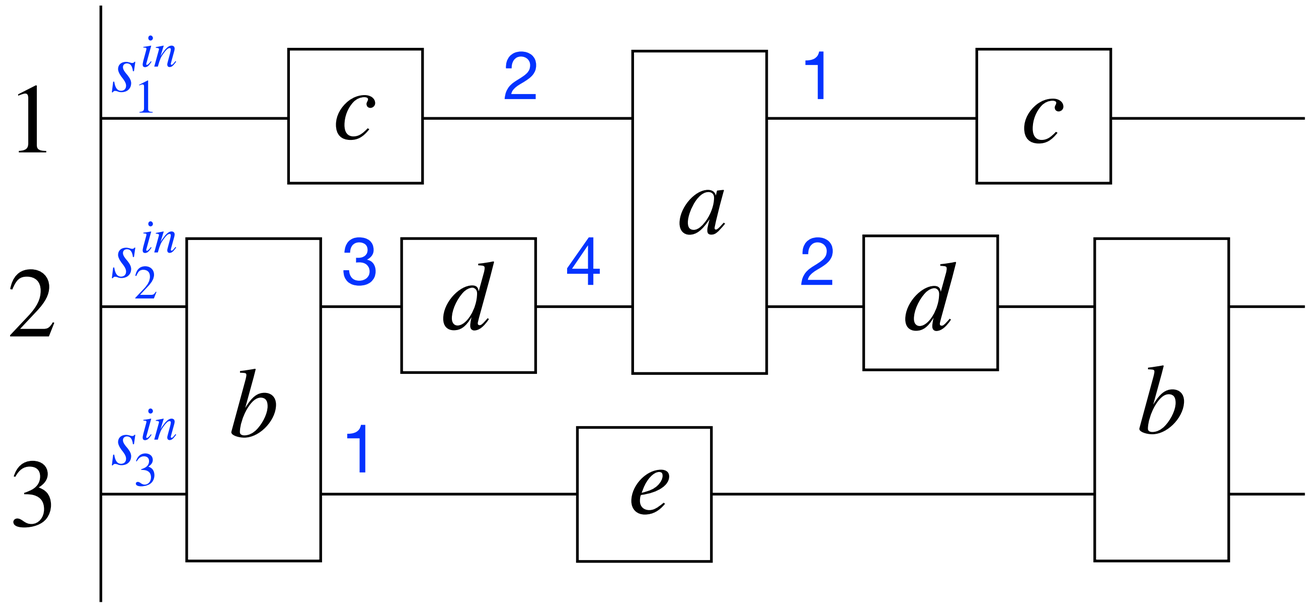


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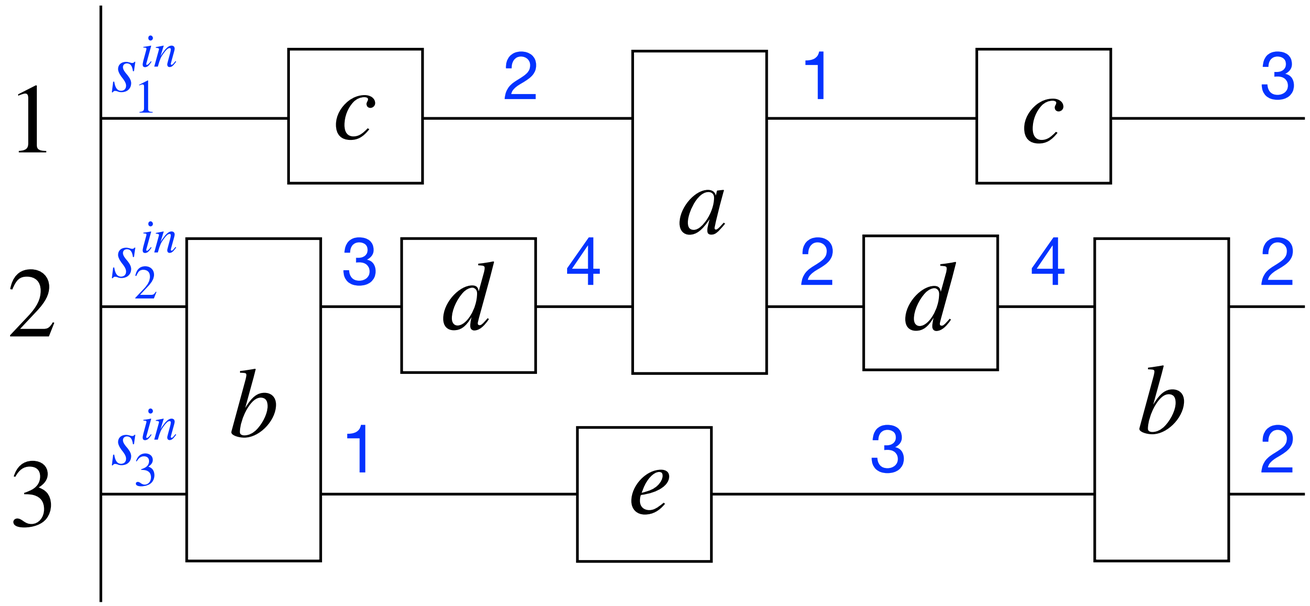


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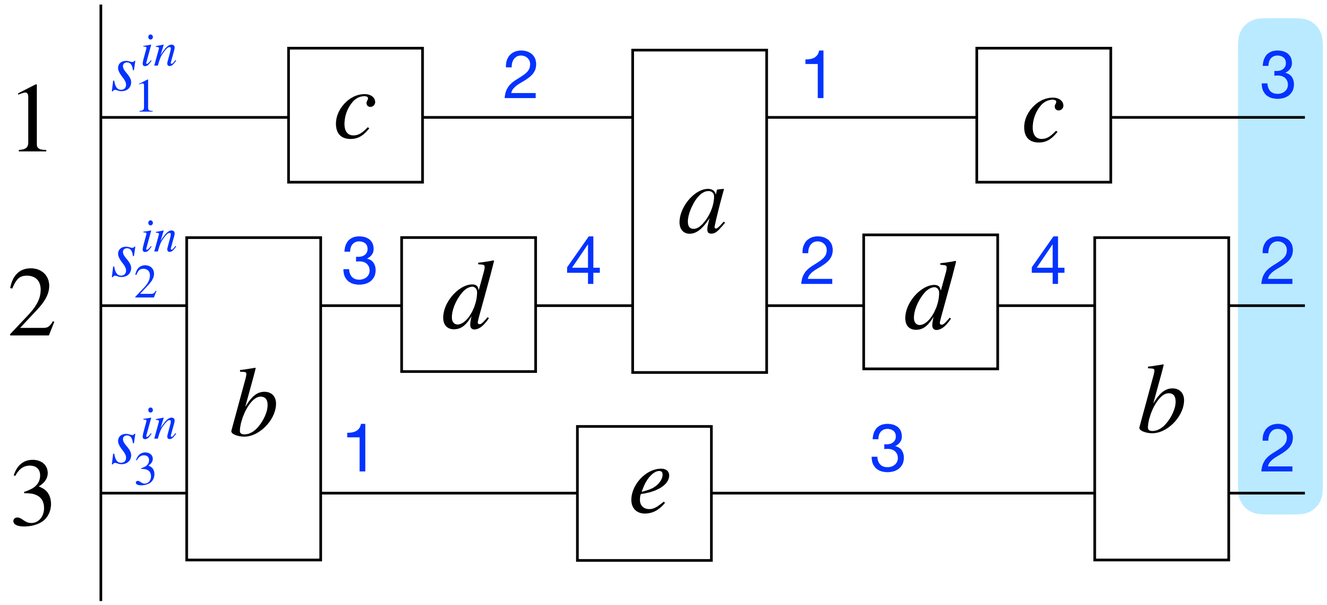


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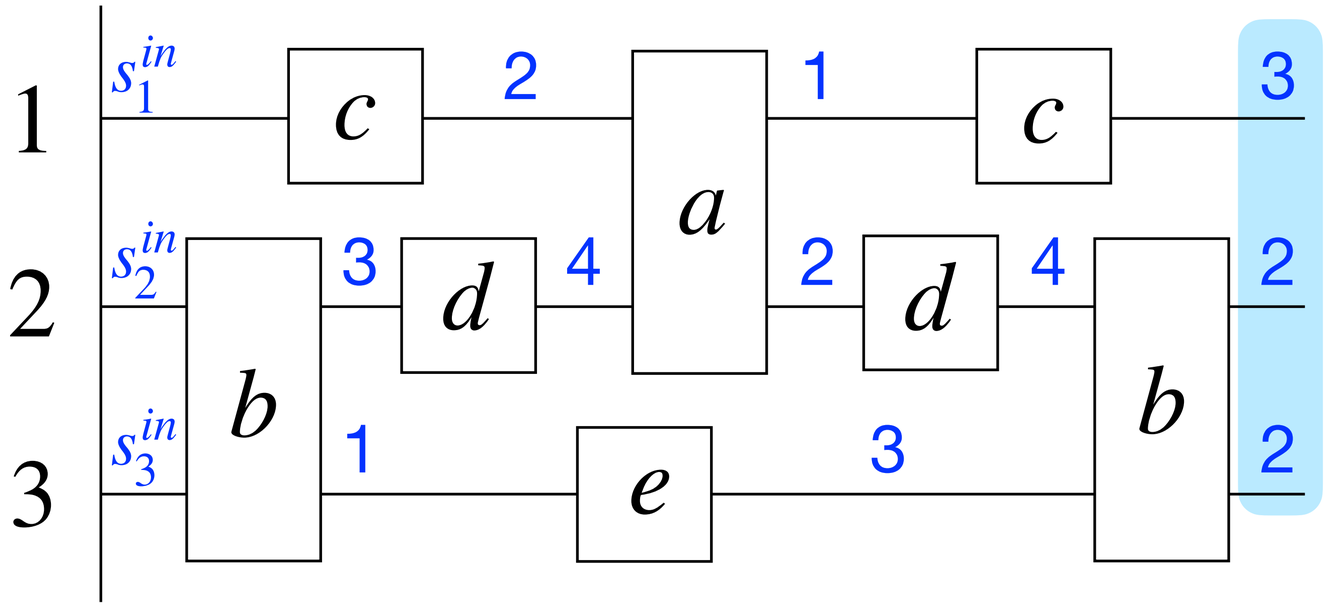


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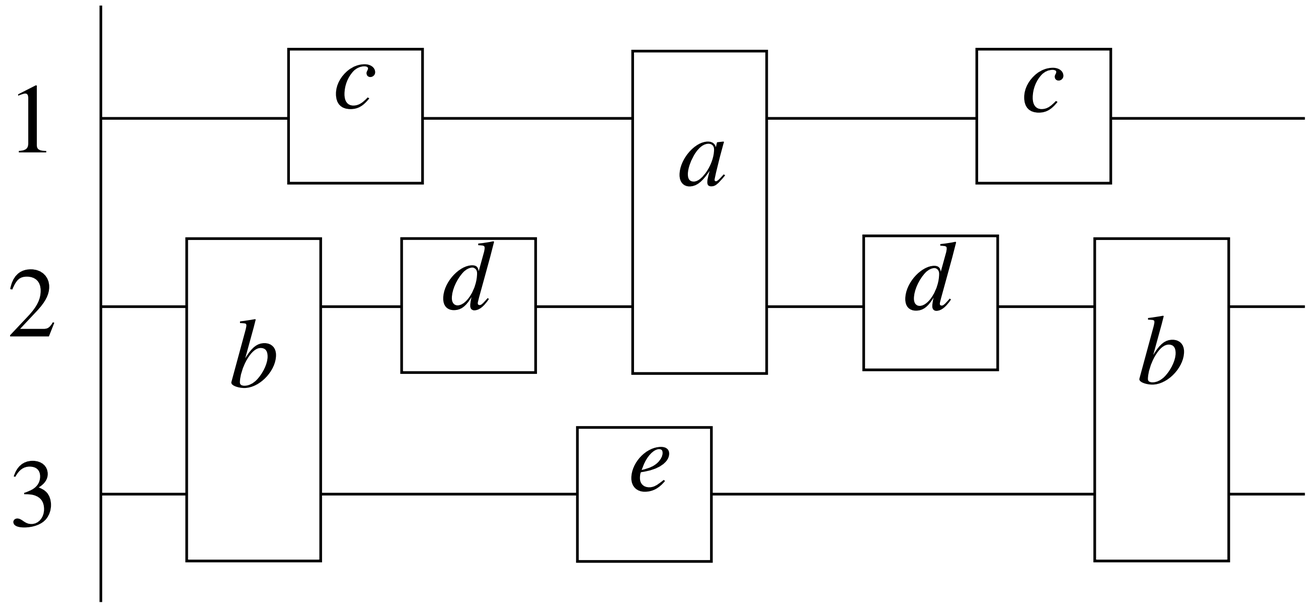
Theorem (Zielonka, 1987)

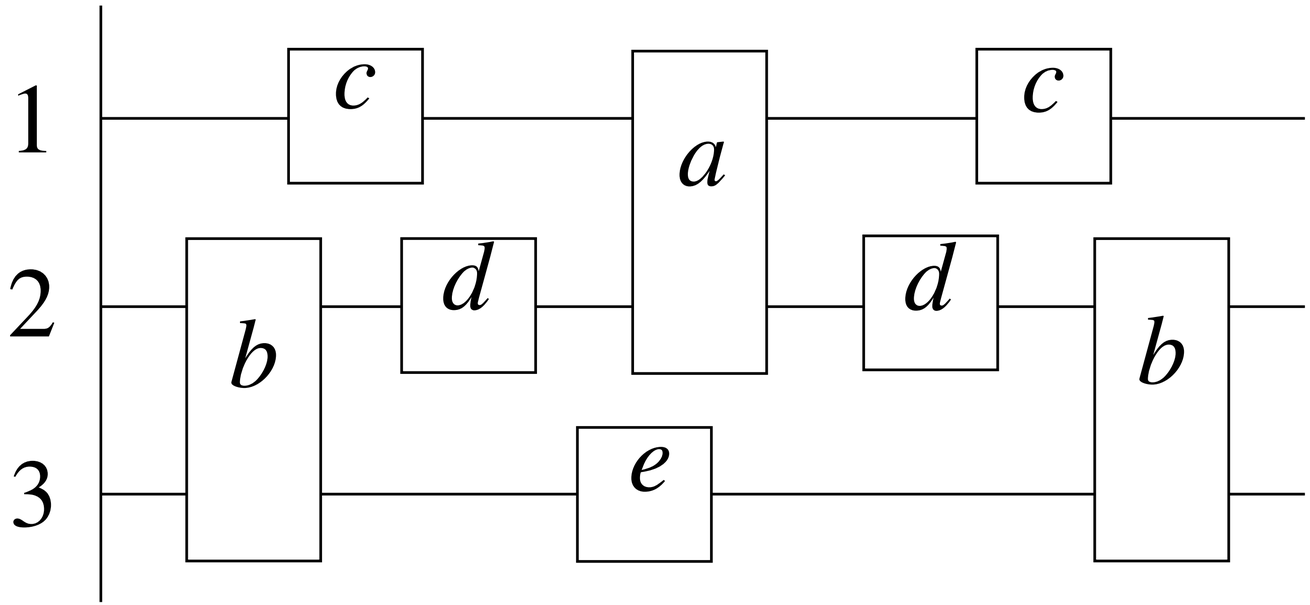
Asynchronous Automata = Regular Trace Languages

# Labeling function




$\theta\colon Tr(\Sigma) \to Tr(\Sigma \times \Gamma)$

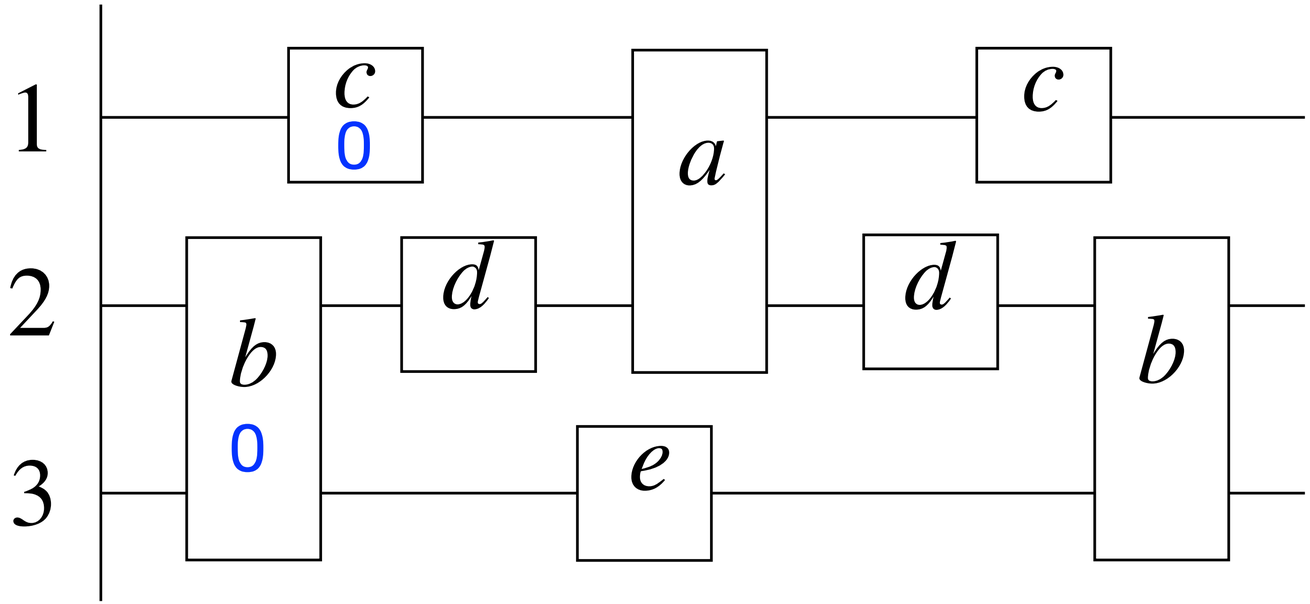
# Labeling function




$\theta: Tr(\Sigma) \rightarrow Tr(\Sigma \times \Gamma)$

$\Gamma = \{0,1\}$ and $Y_3 \leq Y_2$

# Labeling function

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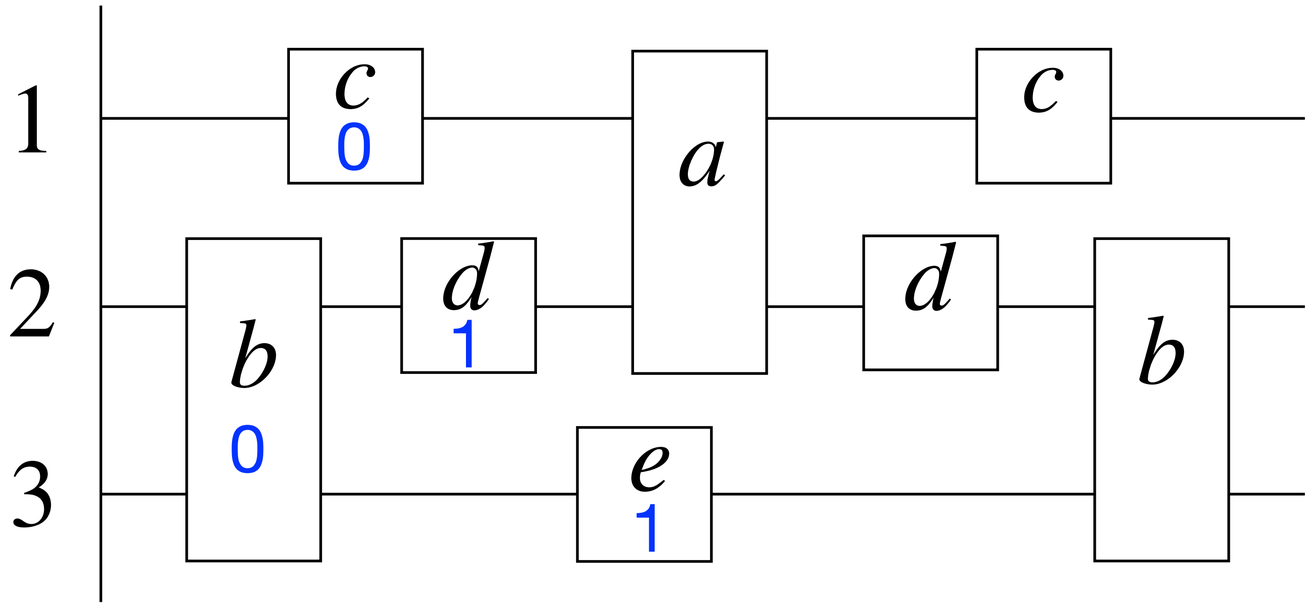


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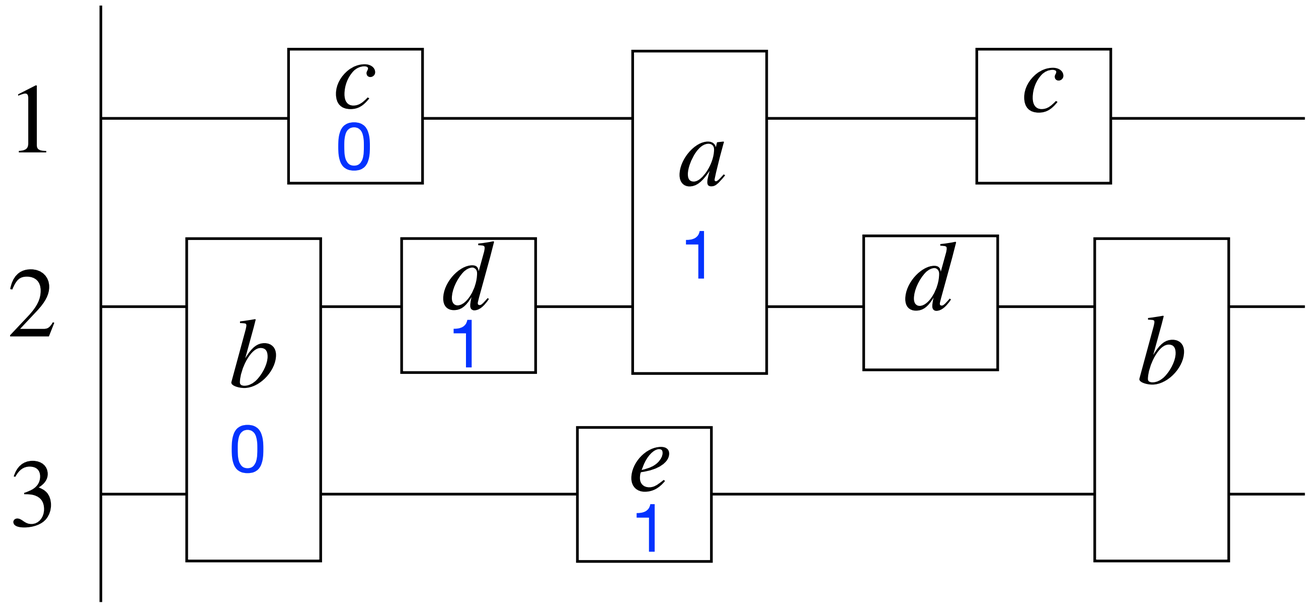


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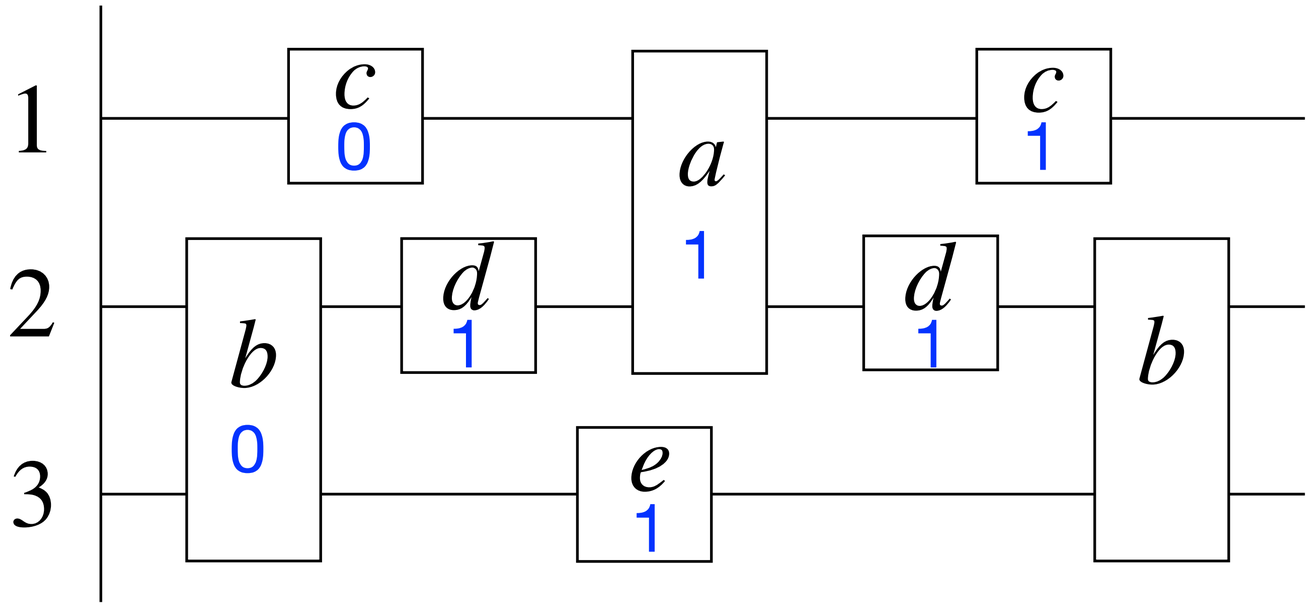


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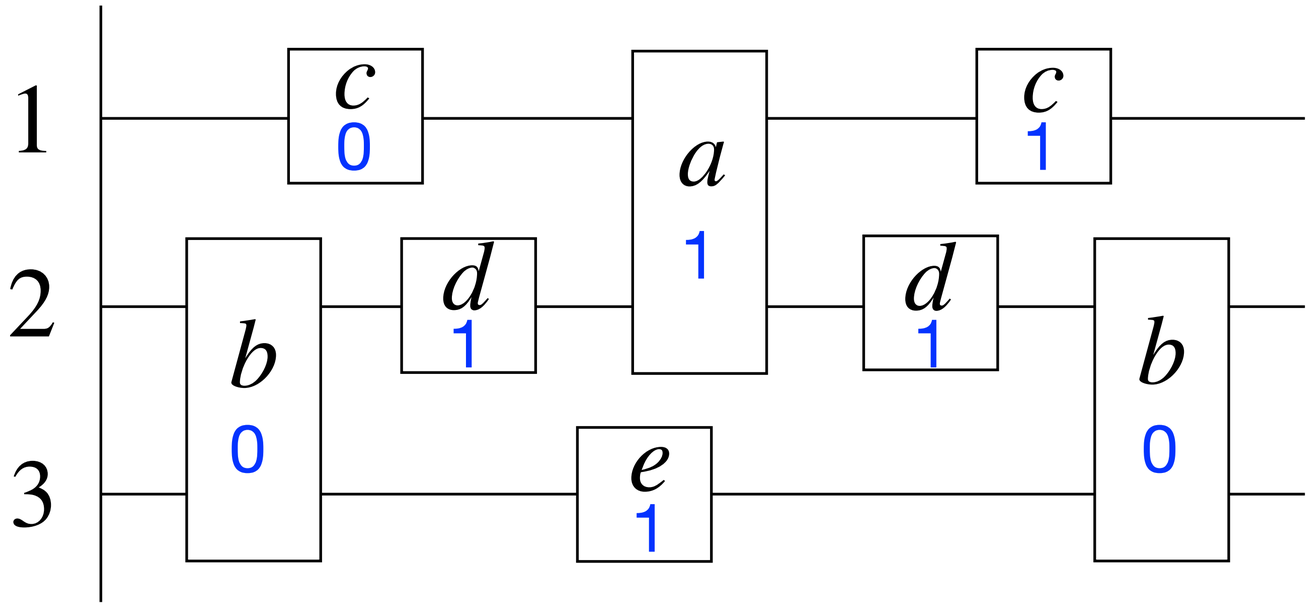


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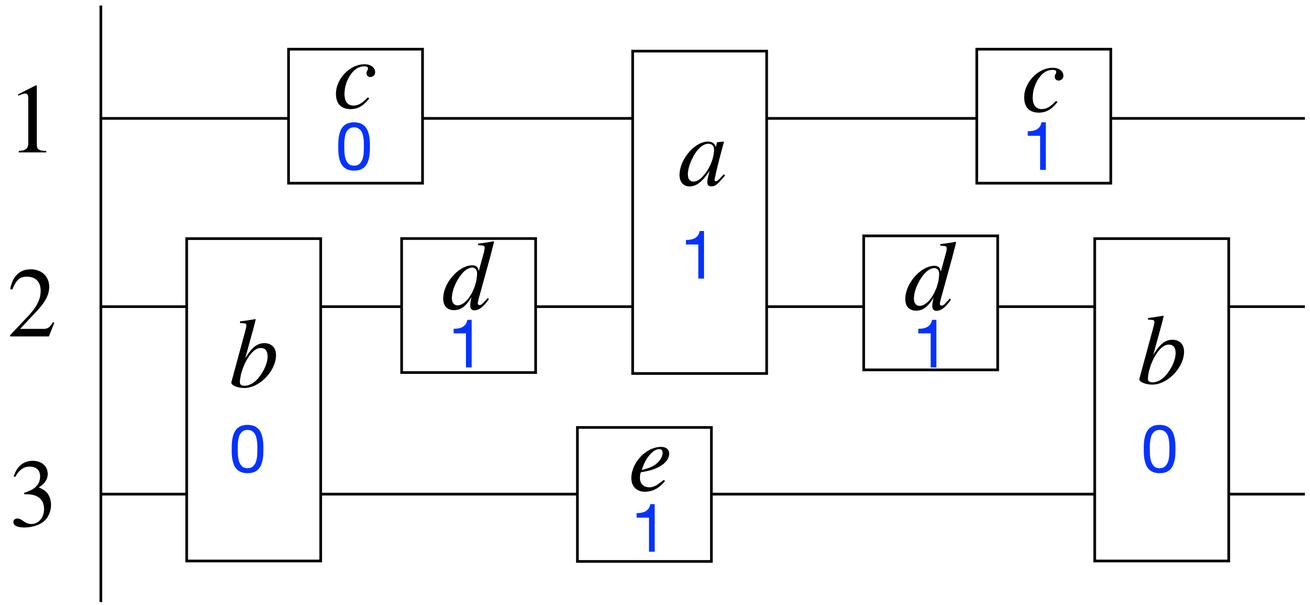
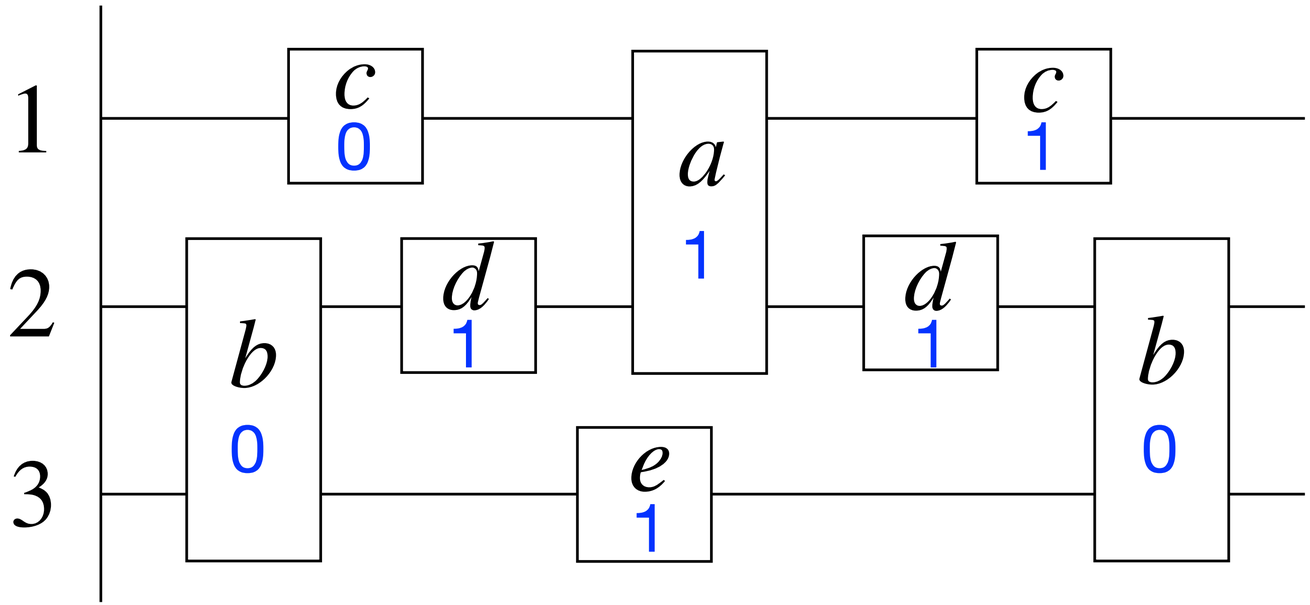



$\theta: Tr(\Sigma) \rightarrow Tr(\Sigma \times \Gamma)$

$\Gamma = \{0,1\}$ and $Y_3 \leq Y_2$

- $\mathcal{F}$ set of event formulas
- $\theta^{\mathcal{F}}: Tr(\Sigma) \rightarrow Tr\left(\Sigma \times \{0,1\}^{\mathcal{F}}\right)$

# Asynchronous Labeling function

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Asynchronous (letter-to-letter) transducer $\mathcal{T} = \left(\mathcal{A}, \{\mu_a\}_{a\in\Sigma}\right)$

- $\mathcal{A} = \left(\{S_i\}_{i\in\mathcal{P}}, \{\delta_a\}_{a\in\Sigma}, s^{in}\right)$
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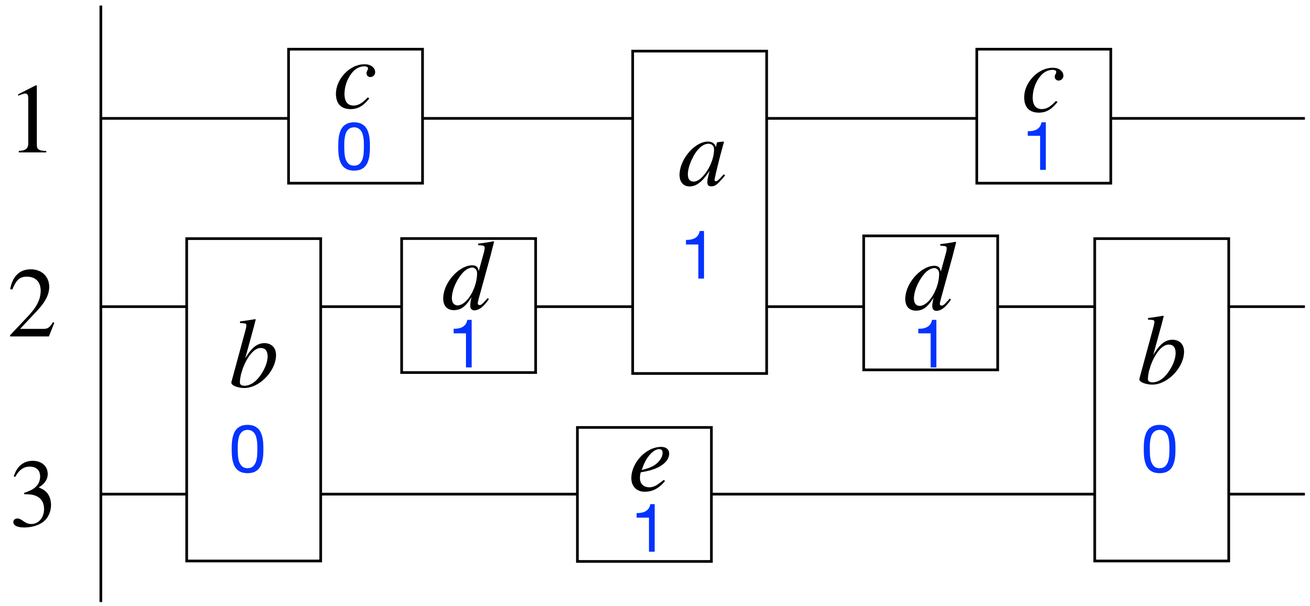


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# Gossip (Asynchronous) Transducer

Theorem (Mukund-Sohoni 1997)

Let $\mathcal{Y} = \{\mathsf{Y}_i \leq \mathsf{Y}_j\,, \mathsf{Y}_{i,k} \leq \mathsf{Y}_j \mid i,j,k \in \mathcal{P}\}$.

There is an asynchronous letter-to-letter transducer $\mathcal{G}$ which computes

$$\theta^{\mathcal{Y}} : Tr(\Sigma) \to Tr\left(\Sigma \times \{0,1\}^{\mathcal{Y}}\right)$$

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- ✅ Model of Concurrency: Mazurkiewicz Traces
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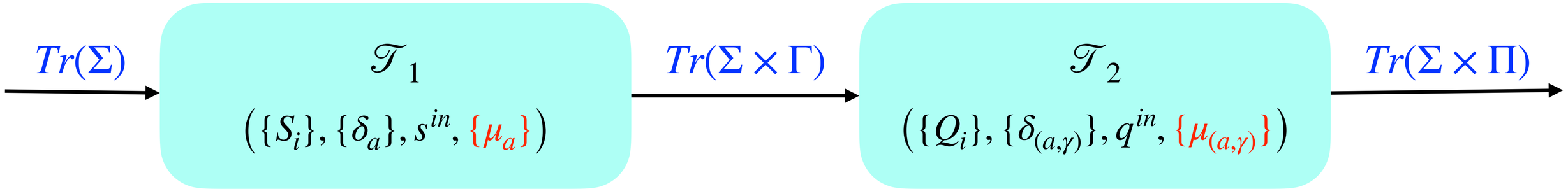

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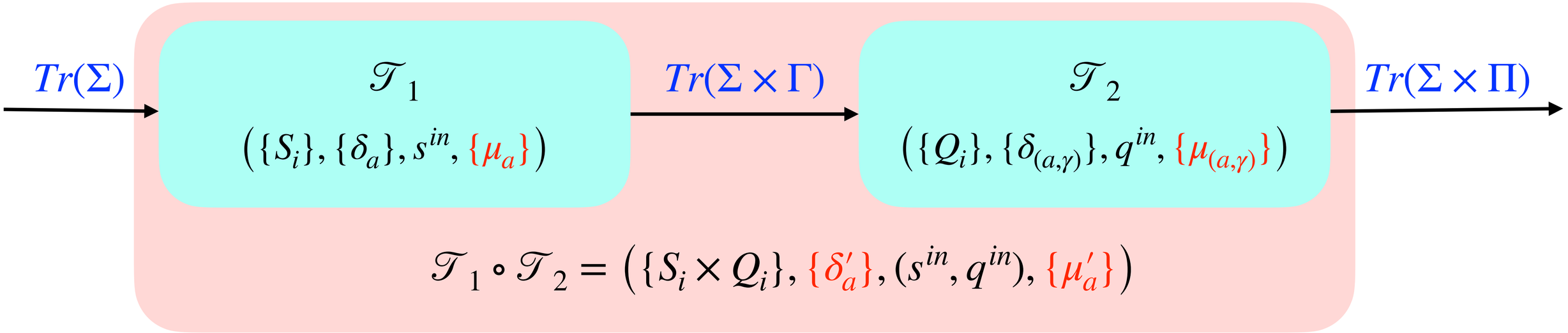

# Cascade Decomposition

Theorem 2

Any regular trace language is accepted by a cascade product of the gossip transducer followed by a sequence of local asynchronous transducers:

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Proof sketch:

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- For each event formula $\varphi$ we construct by structural induction such a cascade product computing $\theta^{\varphi}$

$$\varphi ::= a \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle\pi\rangle\varphi \mid \mathsf{Y}_i \leq \mathsf{Y}_j \mid \mathsf{Y}_{i,k} \leq \mathsf{Y}_j$$

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  - Boolean connectives are trivial
  - Local past Path expressions $\langle\pi\rangle\,\varphi$ are (rather) easy

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Corollary: Zielonka's theorem

Asynchronous Automata = Regular Trace Languages

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Bonus: Using Krohn-Rhodes theorem

Each local asynchronous transducer $\mathscr{T}$ can be chosen to be (on its non-trivial component)

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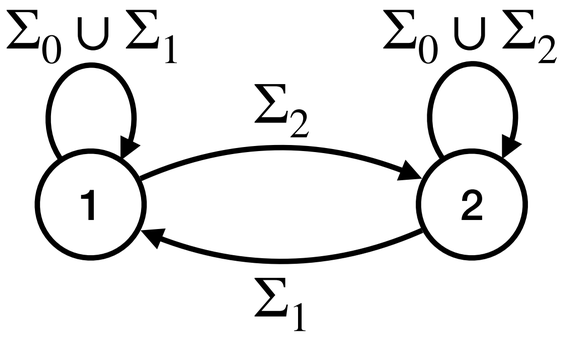

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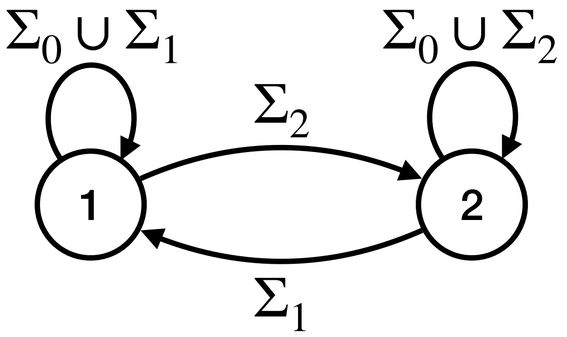
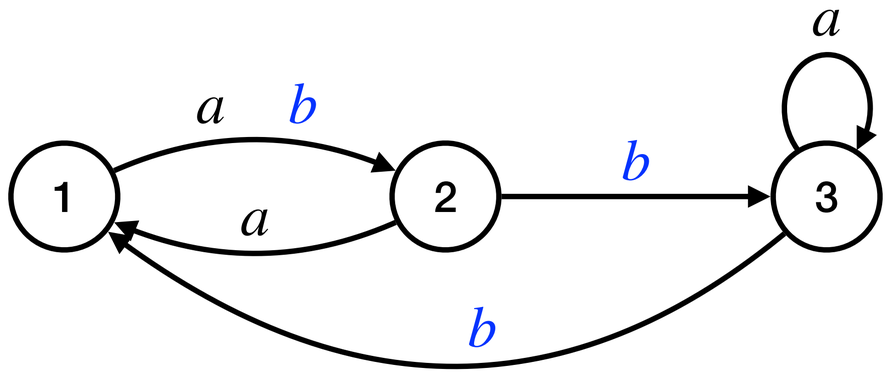



Permutation



# Aperiodic = FO-definable

Theorem [Adsul, Gastin, Sarkar, Weil — Concur'20, LMCS'22]

Any aperiodic (FO) trace language is accepted by a cascade product of the gossip transducer followed by a sequence of local reset transducers:

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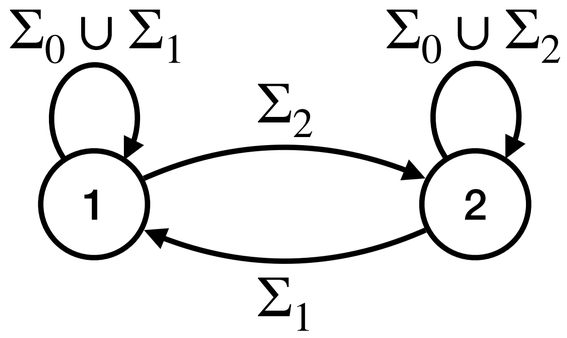

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Direct proof (Not using Krohn-Rhodes theorem)

based on a past temporal logic $\mathrm{LTL}\left(\mathrm{Y}_i \leq \mathrm{Y}_j, \mathrm{S}_i\right)$ proved expressively complete for FO

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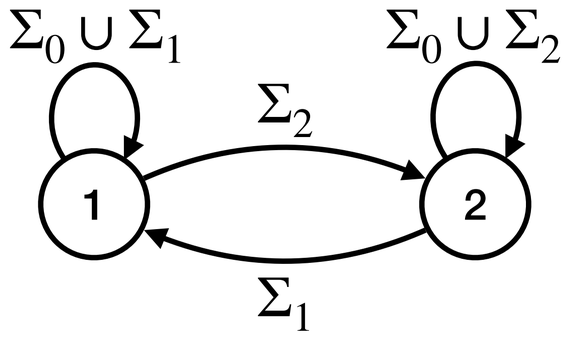

# Outline

- ✓ Model of Concurrency: Mazurkiewicz Traces
- ✓ A propositional dynamic logic for traces
- ✓ Asynchronous automata (Zielonka) and asynchronous labeling functions
- ✓ Cascade product / decomposition (Krohn-Rhodes)
- Conclusion

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- Generalisation to Message Sequence Charts (Message passing automata)?

# Thank you for your attention!