WEIGHTED AUTOMATA HIGHLIGHTED EXCERPTS

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Pieces of choice

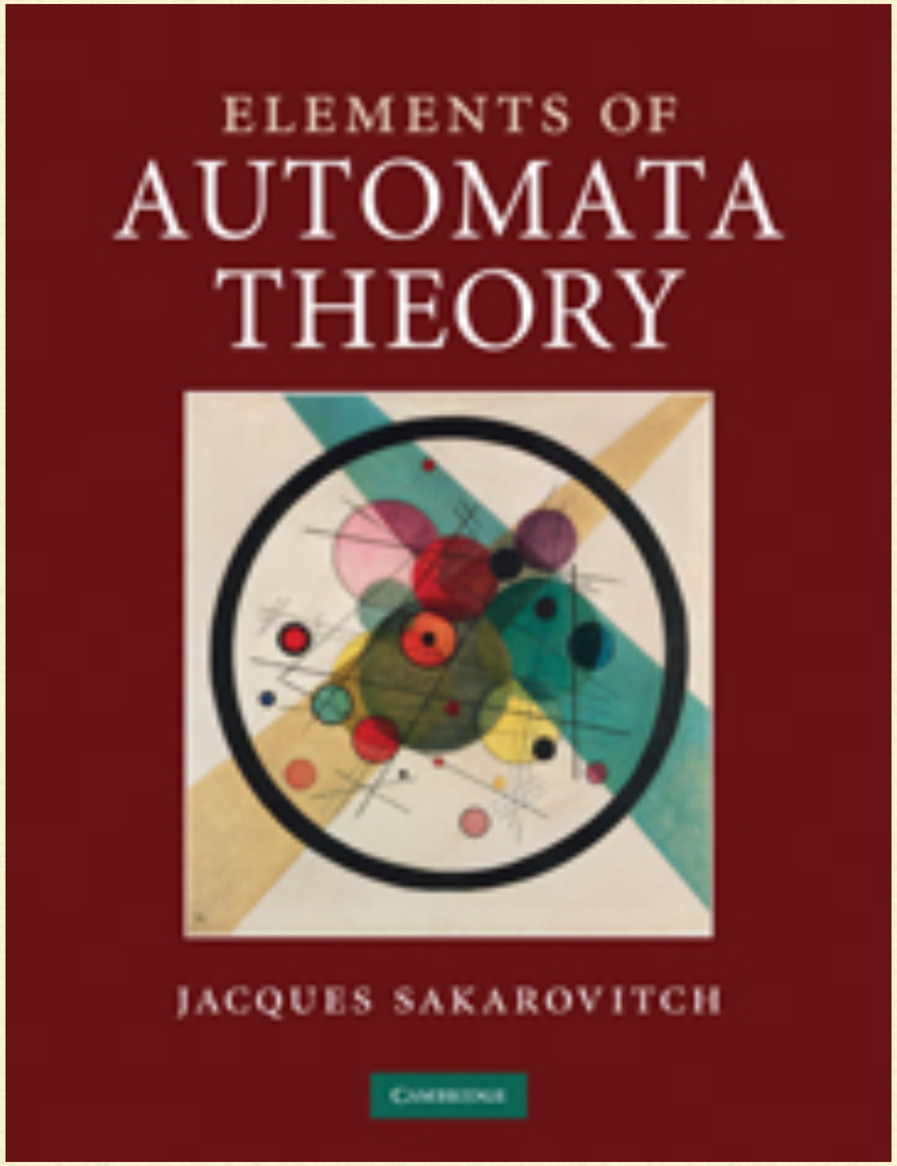
Chosen pieces

GOALS

- Finite representations of functions $F : \Sigma^* \rightarrow S$
- Evaluate such functions given a representation and an input word
- Study/Decide properties of such functions

REFERENCES

REFERENCES



- Lecture notes from MPRI course 2-16

  http://perso.telecom-paristech.fr/~jsaka/ENSG/MPRI/mpri.html

REFERENCES

- Lecture notes from MPRI course 2009, 2010

Weighted Automata

— Version of September 13, 2015 —

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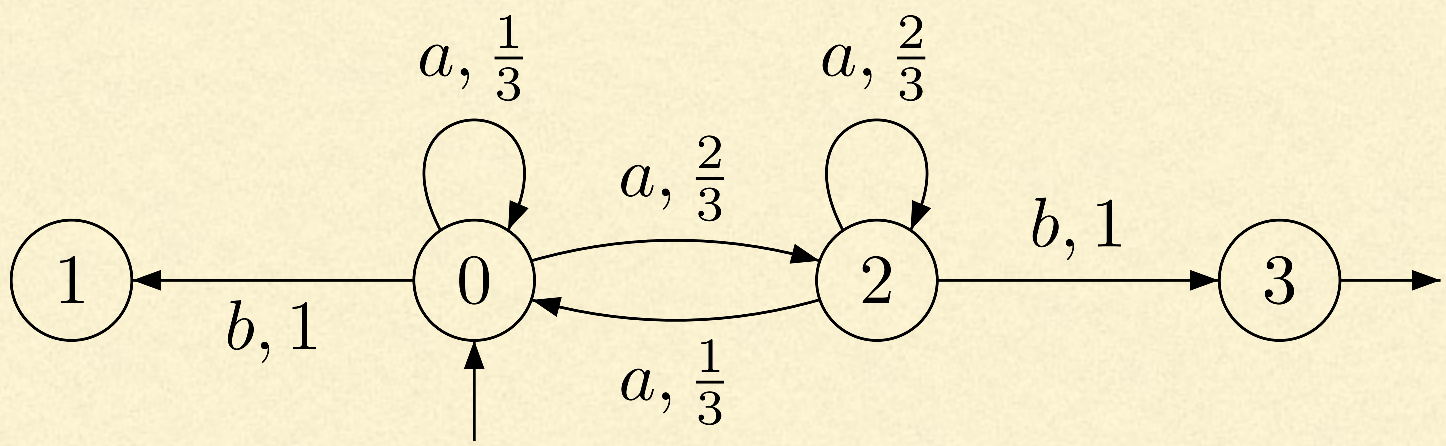
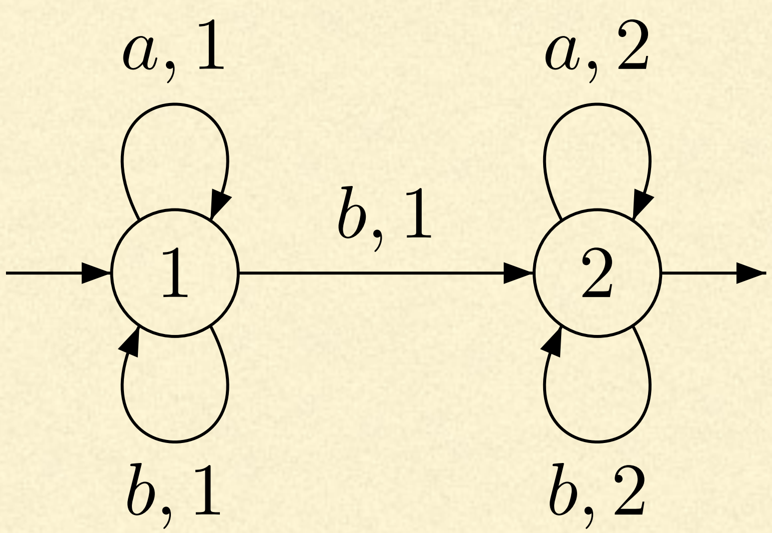
WEIGHTED AUTOMATA

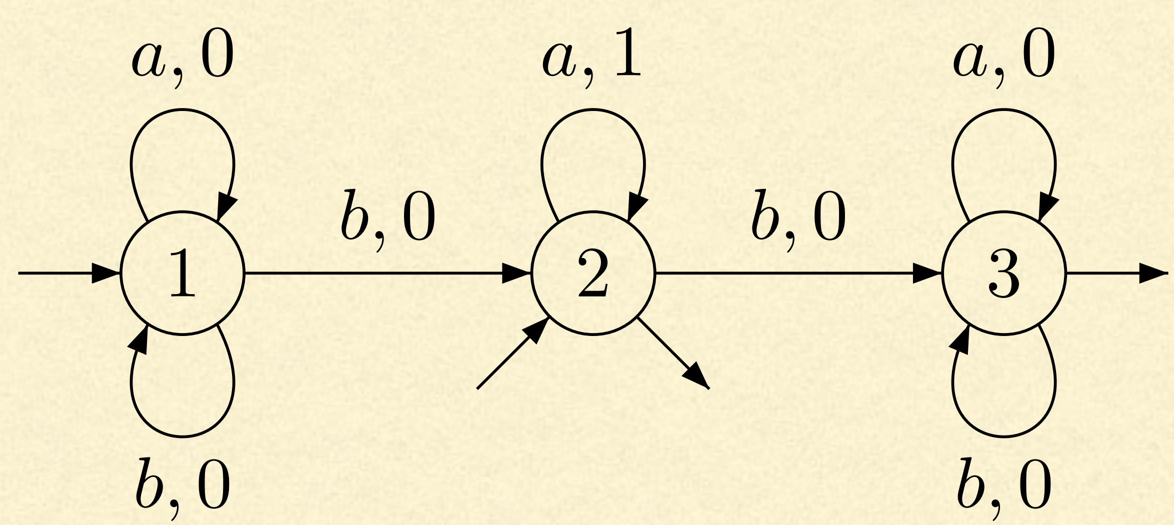
1. Definitions, Examples, Various semantics
2. Boolean vs Quantitative languages
3. Some decision problems
4. Extensions I: Infinite words, Trees, Pictures, Graphs
5. Extensions II: Pebble Walking Automata

R-WEIGHTED AUTOMATA

- R-WA = Automaton + weights from set R



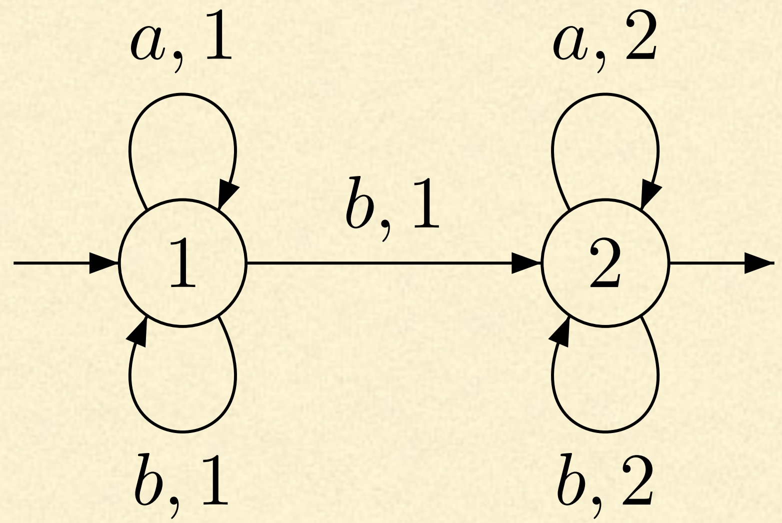




SEMANTICS

- An automaton generates accepting runs ρ on a word w = babaab
- A run ρ generates a sequence of weights: $wgt(\rho) = s_1\ s_2\ s_3\ s_4\ s_5\ s_6$

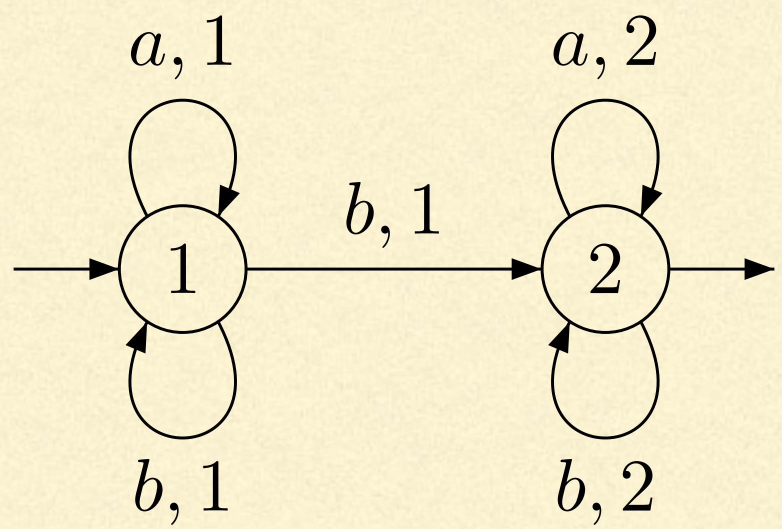
$$\begin{array}{l} \quad\quad\quad\ \ b\quad a\quad b\quad a\quad a\quad b \\ \rho_1 = 1\rightarrow 2\rightarrow 2\rightarrow 2\rightarrow 2\rightarrow 2\rightarrow 2 \\ \rho_2 = 1\rightarrow 1\rightarrow 1\rightarrow 2\rightarrow 2\rightarrow 2\rightarrow 2 \\ \rho_3 = 1\rightarrow 1\rightarrow 1\rightarrow 1\rightarrow 1\rightarrow 1\rightarrow 2 \end{array}$$



$$wgt(\rho_1) = 1\ 2\ 2\ 2\ 2\ 2$$
$$wgt(\rho_2) = 1\ 1\ 1\ 2\ 2\ 2$$
$$wgt(\rho_3) = 1\ 1\ 1\ 1\ 1\ 1$$

SEMANTICS

- An automaton generates accepting runs ρ on a word w = babaab
- A run ρ generates a sequence of weights: wgt(ρ) = $s_1\ s_2\ s_3\ s_4\ s_5\ s_6$
- The value of a weight sequence is computed: Val($s_1\ s_2\ s_3\ s_4\ s_5\ s_6$)



wgt(ρ_1) = 1 2 2 2 2 2
wgt(ρ_2) = 1 1 1 2 2 2
wgt(ρ_3) = 1 1 1 1 1 1

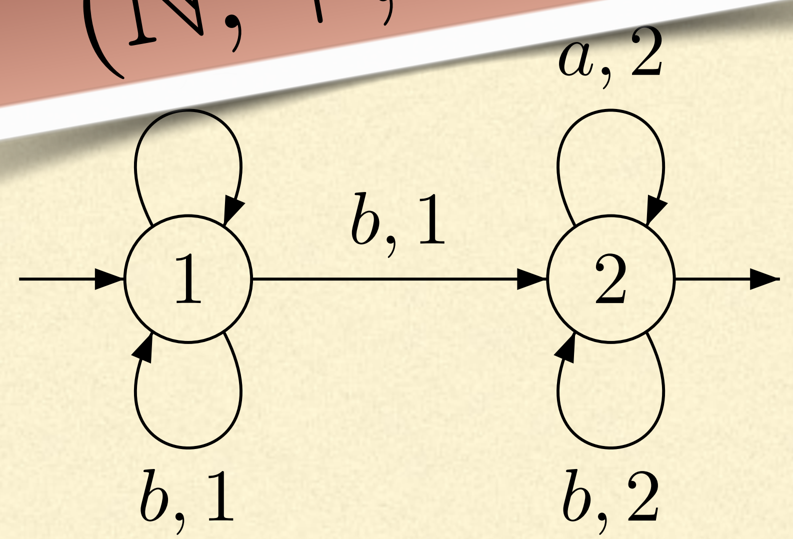
Val(1 2 2 2 2 2) = 2^5
Val(1 1 1 2 2 2) = 2^3
Val(1 1 1 1 1 1) = 2^0

Val = Product

SEMANTICS

- An automaton generates accepting runs ρ on a word w = babaab
- A run ρ generates a sequence of weights: wgt(ρ) = $s_1\ s_2\ s_3\ s_4\ s_5\ s_6$
- The value of a weight sequences is computed: Val($s_1\ s_2\ s_3\ s_4\ s_5\ s_6$)
- Final semantics: $[\![\mathcal{A}]\!](w) = \mathsf{F}\{\{\mathsf{Val}(\mathsf{wgt}(\rho)) \mid \rho \text{ run on } w\}\}$

Natural semiring $(\mathbb{N}, +, \times, 0, 1)$



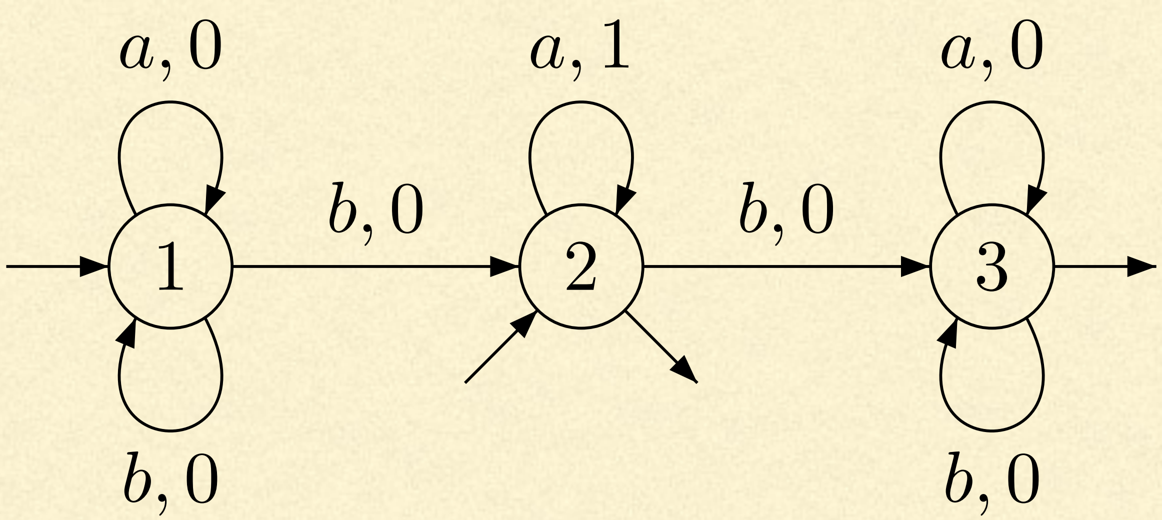
Val(1 2 2 2 2 2) = 2^5
Val(1 1 1 2 2 2) = 2^3
Val(1 1 1 1 1 1) = 2^0

Val = Product

F = Sum

$$[\![\mathcal{A}]\!](babaab) = 2^5 + 2^3 + 2^0 = 41$$

EXAMPLE



(max,+) semiring

$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

w = aababaaab

RUNS	WEIGHTS	VALUES
a a b a b a a a b		
1→1→1→1→1→1→1→1→1→2	0 0 0 0 0 0 0 0 0	0
1→1→1→1→1→2→2→2→2→3	0 0 0 0 0 1 1 1 0	3
1→1→1→2→2→3→3→3→3→3	0 0 0 1 0 0 0 0 0	1
2→2→2→3→3→3→3→3→3→3	1 1 0 0 0 0 0 0 0	2

$$[\![\mathcal{A}]\!](aababaaab) = \max(0,3,1,2) = 3$$

VALUATION FUNCTIONS

- A run generates a sequence of weights $\mathsf{wgt}(\rho) = s_1 s_2 \cdots s_n$
- We compute the value of the sequence $\mathsf{Val}(s_1 s_2 \cdots s_n)$

$\mathsf{Val}\colon S^+ \to S$

- sum $\mathsf{Val}(s_1 s_2 \cdots s_n) = s_1 + s_2 + \cdots + s_n$
- product $\mathsf{Val}(s_1 s_2 \cdots s_n) = s_1 \times s_2 \times \cdots \times s_n$
- average $\mathsf{Val}(s_1 s_2 \cdots s_n) = \frac{s_1 + s_2 + \cdots + s_n}{n}$
- discounted $\mathsf{Val}(s_1 s_2 \cdots s_n) = s_1 + \lambda s_2 + \cdots + \lambda^{n-1} s_n$

FINAL SEMANTICS

- A run generates a sequence of weights $\mathsf{wgt}(\rho) = s_1 s_2 \cdots s_n$
- We compute the value of the sequence $\mathsf{Val}(s_1 s_2 \cdots s_n)$
- Final semantics $[\![\mathcal{A}]\!](w) = \mathsf{F}\{\{\mathsf{Val}(\mathsf{wgt}(\rho)) \mid \rho \text{ run on } w\}\}$

multiset

$$F : \mathbb{N}\langle S \rangle \to S$$

- sum
- min, max
- average

WEIGHTS NEED NOT BE NUMERIC

Weight sequence on w = aababaaab

$x \leftarrow 0; y \leftarrow 0$

$x{+}{+}; y \leftarrow \max(x, y)$

$x{+}{+}; y \leftarrow \max(x, y)$

$x \leftarrow 0$

$x{+}{+}; y \leftarrow \max(x, y)$

$x \leftarrow 0$

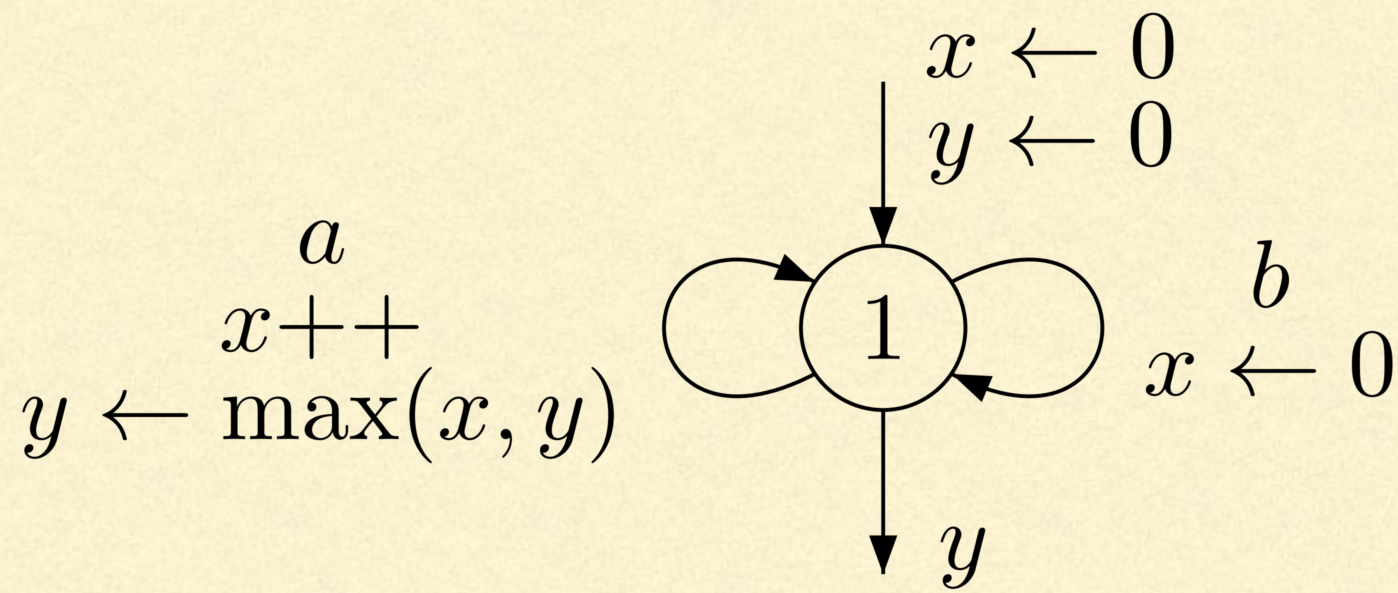
$x{+}{+}; y \leftarrow \max(x, y)$

$x{+}{+}; y \leftarrow \max(x, y)$

$x{+}{+}; y \leftarrow \max(x, y)$

$x \leftarrow 0$

output y



Val = Evaluation of the program

NO FEEDBACK, NO TESTS

- An automaton generates runs
- Runs generate weight sequences and values
- But the computed values do not influence runs: no tests

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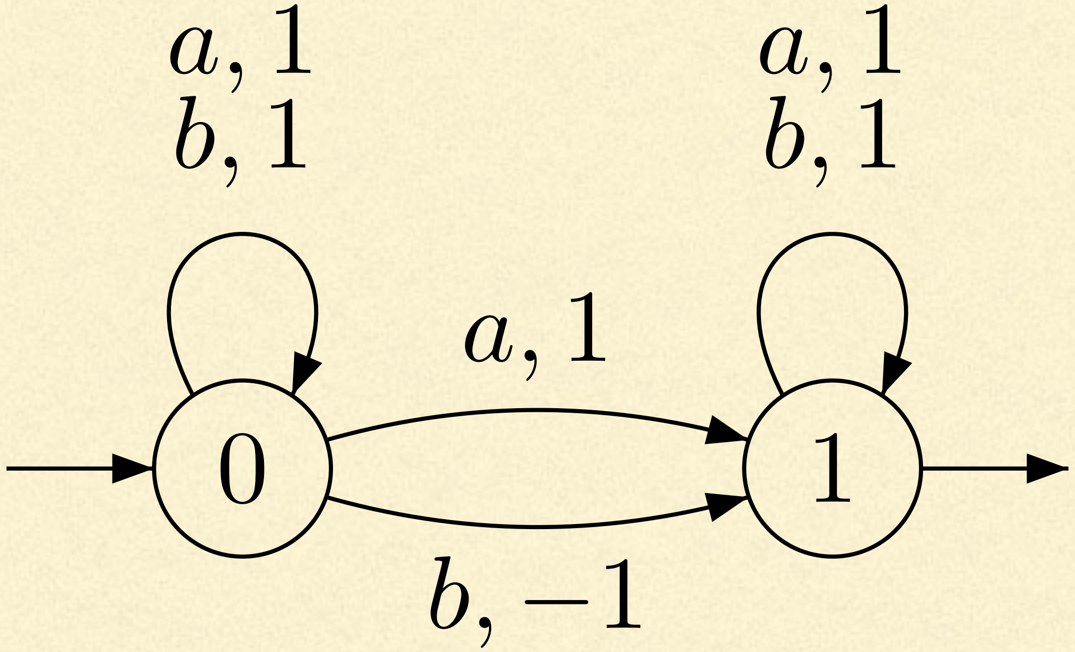
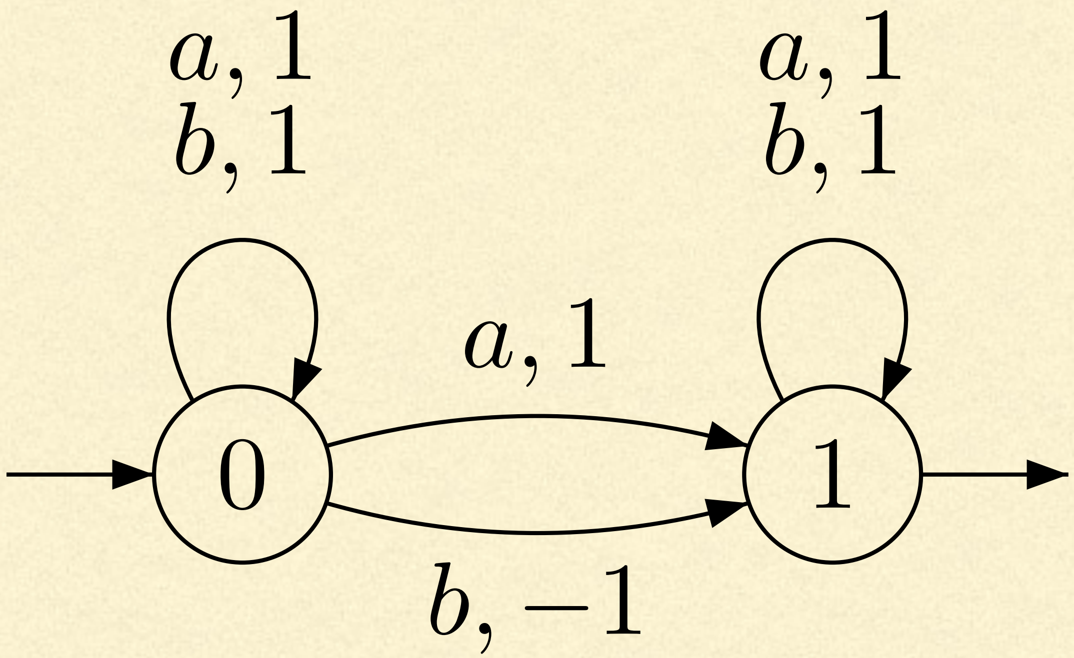
SUPPORT LANGUAGES



$$\mathsf{Support}(\mathcal{A}) = \{w \in \Sigma^+ \mid [\![\mathcal{A}]\!](w) \neq 0\}$$

SUPPORT LANGUAGES

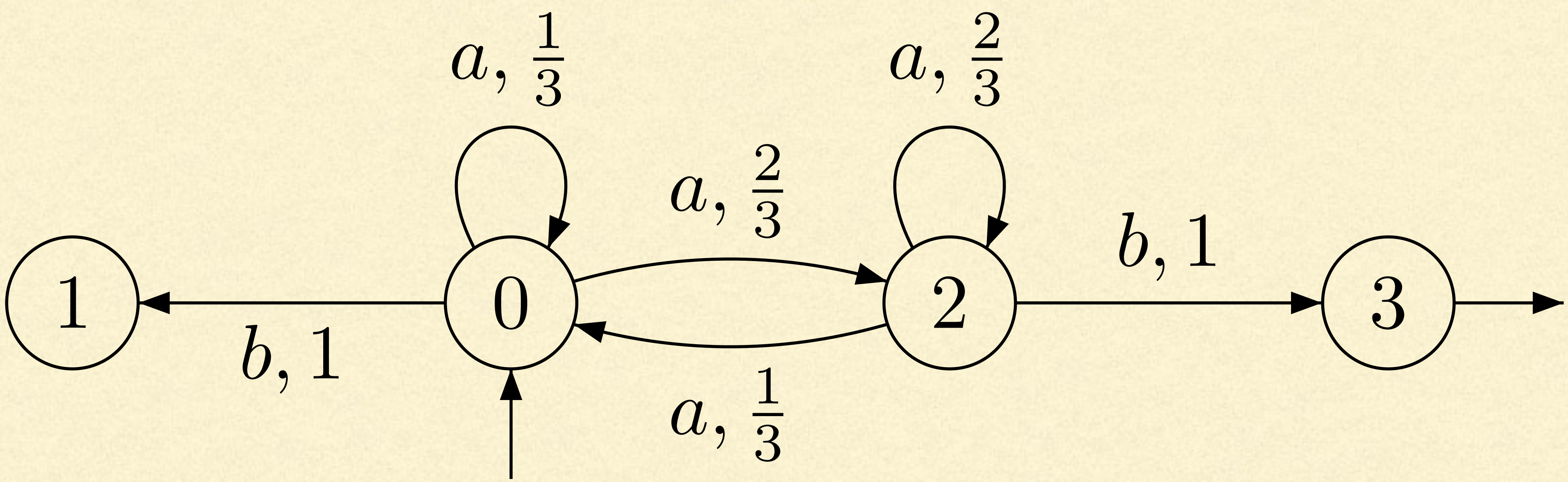


Natural semiring
$(\mathbb{N}, +, \times, 0, 1)$

$$\mathsf{Support}(\mathcal{A}) = \{w \in \Sigma^+ \mid [\![\mathcal{A}]\!](w) \neq 0\}$$

$$\mathsf{Support}(\mathcal{A}) = \{w \in \{a,b\}^+ \mid |w|_a \neq |w|_b\}$$

THRESHOLD LANGUAGES



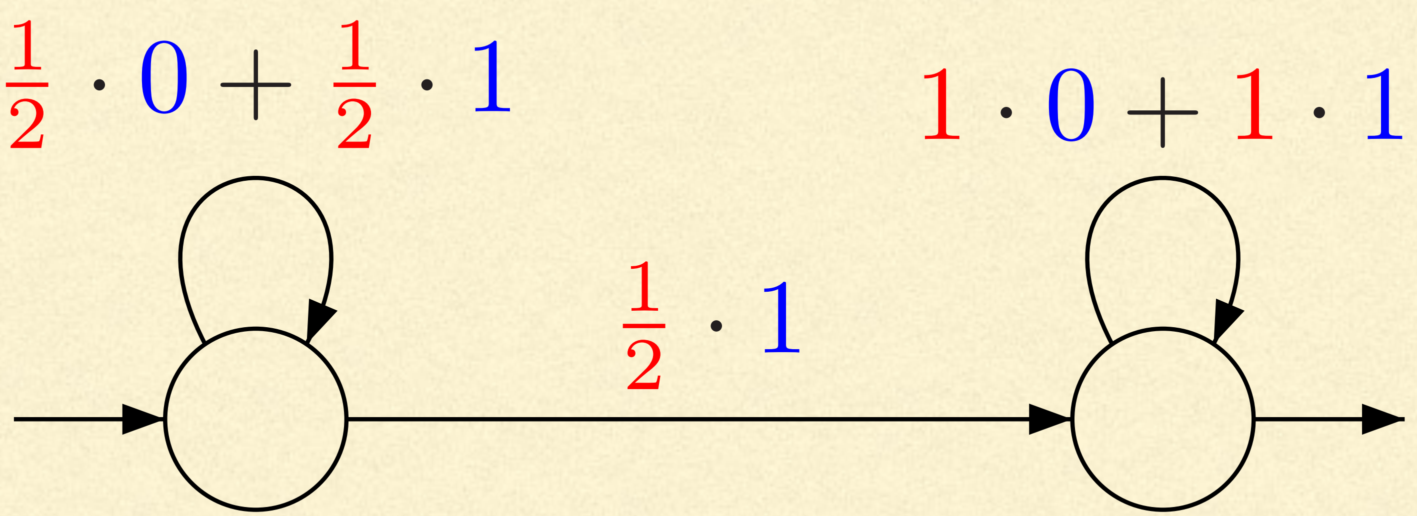
$$\mathcal{L}_{\bowtie\alpha}(\mathcal{A}) = \{w \in \Sigma^+ \mid [\![\mathcal{A}]\!](w) \bowtie \alpha\}$$

$$\bowtie \in \{<, \leqslant, =, \neq, \geqslant, >\}$$

THRESHOLD LANGUAGES

$$\mathcal{L}_{\bowtie\alpha}(\mathcal{A}) = \{w \in \Sigma^+ \mid [\![\mathcal{A}]\!](w) \bowtie \alpha\}$$

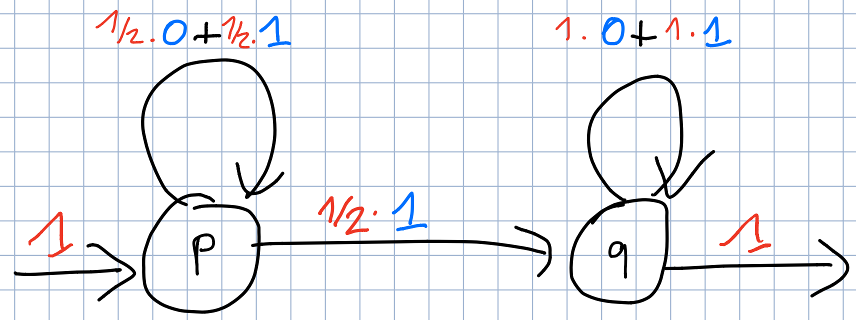
Thm [Rabin 1963]
There are threshold languages that are not
recursively enumerable



Alphabet $A = \{0, 1\}$ Weights: $(\mathbb{R}_{\geq 0}, +, \times, 0, 1)$

Given $w = a_1 a_2 \cdots a_n$ compute the decimal value $0.a_1 a_2 \cdots a_n$ in base 2, i.e., $\frac{a_1}{2} + \frac{a_2}{4} + \cdots + \frac{a_n}{2^n} = \overline{0.w}^2$

$+$: non determinism



Let $w = 01011 \in A^*$ $[\![\mathcal{A}]\!](w) = \overline{0.w}^2 \in [0, 1]$

Runs	Value
$p \xrightarrow[1/2]{0} p \xrightarrow[1/2]{1} q \xrightarrow[1]{0} q \xrightarrow[1]{1} q \xrightarrow[1]{1} q$	$\frac{1}{2^2}$
$p \xrightarrow[1/2]{0} p \xrightarrow[1/2]{1} p \xrightarrow[1/2]{0} p \xrightarrow[1/2]{1} q \xrightarrow[1]{1} q$	$\frac{1}{2^4}$
$p \xrightarrow[1/2]{0} p \xrightarrow[1/2]{1} p \xrightarrow[1/2]{0} p \xrightarrow[1/2]{1} p \xrightarrow[1/2]{1} q$	$\frac{1}{2^5}$

Threshold Language

Let $\alpha \in [0,1]$. Define $L_\alpha = \{ w \in A^* \mid [\![\mathcal{A}]\!](w) > \alpha \}$

Thm: Uncountably many threshold Languages.

$0 < \alpha < \beta < 1 \Rightarrow L_\beta \subsetneq L_\alpha$

Let $w \in A^*$ s.t. $\alpha < \overline{0.w}^2 < \beta$

We have $w \in L_\alpha \setminus L_\beta$ $\square$

Cor: Some Threshold Languages are not recursively enumerable.

THRESHOLD LANGUAGES

$$\mathcal{L}_{\bowtie\alpha}(\mathcal{A}) = \{w \in \Sigma^+ \mid [\![\mathcal{A}]\!](w) \bowtie \alpha\}$$

Thm [Rabin 1963]
There are threshold languages that are not recursively enumerable

Thm [Rabin 1963]
If the threshold is isolated then the threshold language is regular

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DECISION PROBLEMS

- Emptiness

$$\mathsf{Support}(\mathcal{A}) = \varnothing$$

The emptiness problem is decidable in

$$(\mathbb{B}, \vee, \wedge, 0, 1) \qquad (\mathbb{N}, +, \times, 0, 1)$$

$$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$$

$$(\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$$

DECISION PROBLEMS

- Emptiness $\mathsf{Support}(\mathcal{A}) = \varnothing$

Reachability in graphs
NLOGSPACE

The emptiness problem is

$$(\mathbb{B}, \vee, \wedge, 0, 1) \qquad (\mathbb{N}, +, \times, 0, 1)$$

$$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$$

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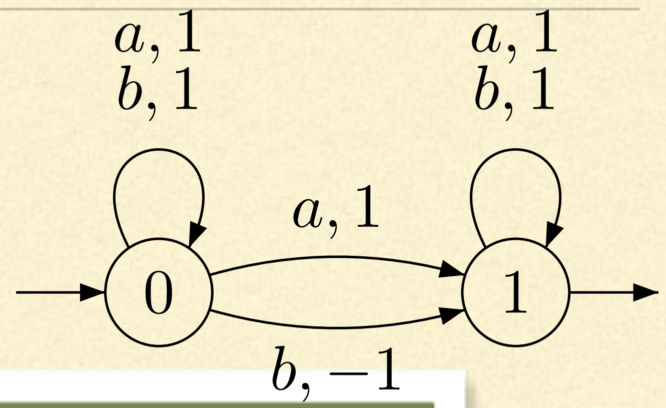
DECISION PROBLEMS

- Emptiness

$$\mathsf{Support}(\mathcal{A}) = \varnothing$$



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$$(\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$$

$$(\mathbb{Z}, +, \times, 0, 1) \qquad (\mathbb{Q}, +, \times, 0, 1)$$

DECISION PROBLEMS

- Given $\mathcal{A} = (Q, \Sigma, \Delta, I, F, \text{wgt})$ $(\mathbb{Z}, +, \times, 0, 1)$

The following problems are undecidable

$$[\![\mathcal{A}]\!](w) = 0 \qquad [\![\mathcal{A}]\!](w) > 0$$

1. for some word w
2. for infinitely many words w

DECISION PROBLEMS

- Support recognizability:

Kirsten-Quass 2011: The support recognizability problem is undecidable in

$$(\mathbb{Z}, +, \times, 0, 1)$$

Kirsten 2011: Over zero-sum free commutative semirings, the support is always recognizable

$$(\mathbb{B}, \vee, \wedge, 0, 1) \qquad (\mathbb{N}, +, \times, 0, 1)$$

$$(\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$$

$$(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$$

DECISION PROBLEMS

- Equivalence $$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$$

The equivalence problem is decidable in

$$(\mathbb{B}, \vee, \wedge, 0, 1) \qquad (\mathbb{N}, +, \times, 0, 1)$$
$$(\mathbb{Z}, +, \times, 0, 1) \qquad (\mathbb{Q}, +, \times, 0, 1)$$

any sub-semiring of a field
(possibly non commutative)

Krob 1992: The equivalence problem is undecidable in

$$(\mathbb{N} \cup \{+\infty\}, \min, +, +\infty, 0)$$

Tropical Semiring $\text{Trop}_{\mathbb{N}} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$

$\text{Trop}_{\mathbb{Z}} = (\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$

Input: $\mathcal{A}$, $\mathcal{B}$ over Trop.

Question: $[\![\mathcal{A}]\!] \bowtie [\![\mathcal{B}]\!]$? $\bowtie$: $=, \neq, <, \leq$

Krob '93: Undecidable.

Proof below from Th. Colcombet.

Reduction of Halting problem for 2-counter machines.

From a Minsky machine $\mathcal{M}$ we build $\mathcal{A}$ over $\text{Trop}_{\mathbb{Z}}$ s.t.
$\mathcal{M}$ does not halt iff $[\![\mathcal{A}]\!](w) \leq -1 \quad \forall\, w \in \Sigma^*$

Encoding: $\Sigma = \Delta \uplus \{a, b\}$

Configuration $C = (q, k, \ell)$ $\tilde{C} = a^k b^\ell$

Computation $C_0 \xrightarrow{\delta_1} C_1 \xrightarrow{\delta_2} C_2 \cdots$ $\tilde{C}_0\, \delta_1\, \tilde{C}_1\, \delta_2\, \tilde{C}_2 \cdots$

We build $\mathcal{A}$ s.t.

$[\![\mathcal{A}]\!](w) = 0$ iff w encodes an halting computation!

$[\![\mathcal{A}]\!](w) \leq -1$ otherwise i.e. for all incorrect words

A word $w \in \Sigma^*$ is incorrect if

1) not in $a^*b^*(\Delta a^*b^*)^*$
2) Does not start in $C_0 = (q_0, 0, 0)$
3) Does not end in q_f
4) Non Consecutive Transitions
5) Illegal zero test
6) Wrong increment on c_1
7) Wrong decrement on c_1
8) Wrong change of c_1
9) idem on c_2

5) No factor of the form

$$a b^* (p \xrightarrow{c_1=0?} q)$$

or

$$b (p \xrightarrow{c_2=0?} q)$$

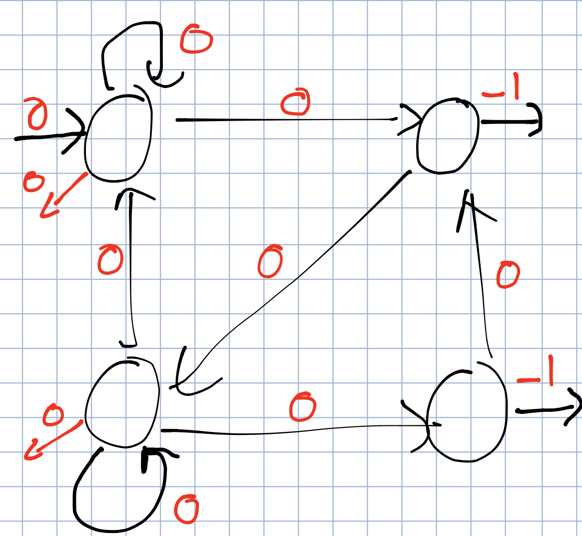
(1-5) are rational constraints

Let $L \subseteq \Sigma^*$ be the rational language for (1-5)

Let $\mathcal{B}$ be a complete DFA recognizing L

Add weights

- 0 on all transitions
- 0 on the initial state
- -1 on final states
- 0 on <u>non</u> final states

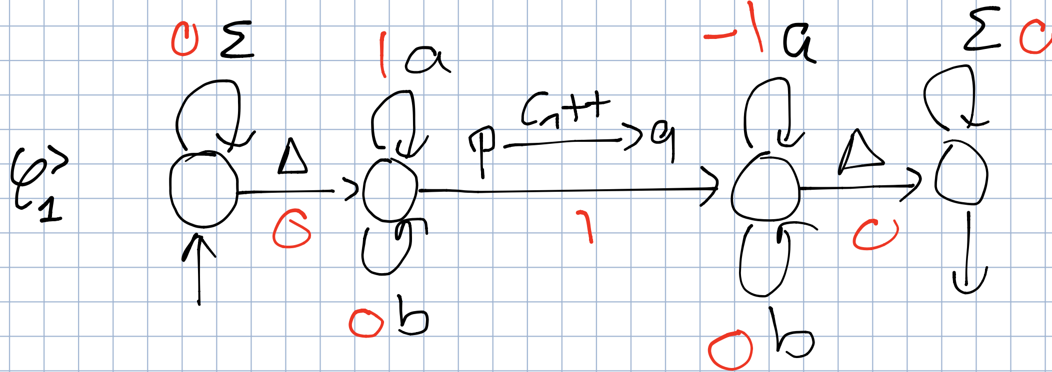


$$[\![\mathcal{B}]\!](w) = \begin{cases} -1 & \text{if } w \in L, \text{ i.e., } w \models 1 \vee 2 \vee 3 \vee 4 \vee 5 \\ 0 & \text{otherwise} \end{cases}$$

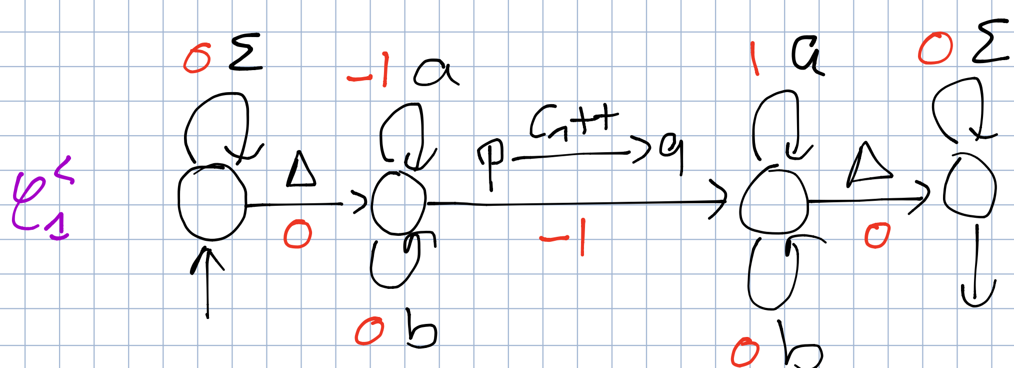
6) Wrong increment on c_1

On $w \in \Sigma^* \Delta a^k b^* (p \xrightarrow{c_1{++}} q)\, a^\ell b^* \Delta \Sigma^*$

$[\![\varphi_1^{>}]\!](w) = k+1-\ell$ <0 if $\ell > k+1$



$[\![\varphi_1^{<}]\!](w) = \ell-(k+1)$ <0 if $\ell < k+1$



Let $\varphi_1^{++} = \varphi_1^{>} \uplus \varphi_1^{<}$ $[\![\varphi_1^{++}]\!](w) = -|\ell-(k+1)| = \begin{cases} 0 & \text{if } \ell = k+1 \\ \leq -1 & \text{otherwise} \end{cases}$

7) Wrong decrement on c_1 Similar.

8) Wrong change of c_1 Similar.

i.e. a transition $\delta \in D$ which does not change c_1 but

$\ldots \Delta a^k b^* \delta a^\ell b^* \Delta \ldots$ with $k \neq \ell$

$\mathcal{A} = \mathcal{B} \uplus \mathcal{C}_1^{++} \uplus \mathcal{C}_1^{--} \uplus \mathcal{C}_1^{=} \uplus \cdots$

$[\![\mathcal{A}]\!] = \min\left([\![\mathcal{B}]\!], [\![\mathcal{C}_1^{++}]\!], \cdots\right)$

Reduction of Inequality to equality

$f \leq g$ iff $f = \min(f, g)$

If $\mathcal{A}_f$ computes f and $\mathcal{A}_g$ computes g then

$\mathcal{A}_f \uplus \mathcal{A}_g$ computes $\min(f, g)$ ✓

Reduction from $\text{Trop}_{\mathbb{Z}}$ to $\text{Trop}_{\mathbb{N}}$

Let $\mathcal{A}$ over $\text{Trop}_{\mathbb{Z}}$ which computes f

For $k \in \mathbb{N}$ Define $\mathcal{A}^{+k}$ by adding k to all weights

Then $[\![\mathcal{A}^{+k}]\!](w) = [\![\mathcal{A}]\!](w) + k \times |w|$ Actually $k \times (|w| + 2)$

Hence $[\![\mathcal{A}]\!] \bowtie [\![\mathcal{B}]\!]$ iff $[\![\mathcal{A}^{+k}]\!] \bowtie [\![\mathcal{B}^{+k}]\!]$ ✓

DECISION PROBLEMS

- Emptiness of a threshold language $\mathcal{L}_{\bowtie\alpha}(\mathcal{A}) = \varnothing$

The value problem is undecidable in

$$(\mathbb{Q}_{\geqslant 0}, +, \times, 0, 1)$$

Input: Prob. Automaton $\mathcal{A}$

Question: $\exists w \in \Sigma^* \quad [\![\mathcal{A}]\!](w) = 1/2$?

Undecidability Paz 71

Proof below: Blondel & Canterini, Gimbert & Oualhadj

Reduction of the PCP problem to the value problem

Lemma 1: Let $\varphi : A^* \to \{0,1\}$ be a morphism

We build $\mathcal{A}_\varphi$ which computes $\overline{0.\varphi(w)}^2 \quad \forall w \in A^+$

Proof: See below

Lemma 2: Let $\psi : A^* \to \{0,1\}$ be a morphism

We build $\mathcal{A}_{1-\psi}$ which computes $1 - \overline{0.\psi(w)}^2 \quad \forall w \in A^+$

Proof: See below

PCP: Given $\varphi, \psi : A^* \to \{0,1\}$ 2 morphisms

$\exists w \in A^+ : \varphi(w) = \psi(w)$?

wlog. $\varphi(A) \subseteq \{0,1\}^*.1 \quad \psi(A) \subseteq \{0,1\}^*.1$

We may add 1 at each even position: $\varphi(a) = 10 \to \varphi'(a) = 1101$

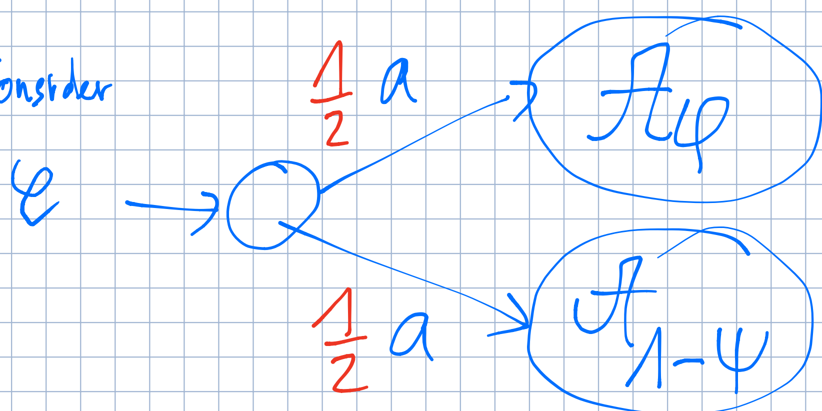
Let $\varphi, \psi : A^* \longrightarrow \{0,1\}^*$ be morphisms defining the PCP problem with $\varphi(A) \cup \psi(A) \subseteq \{0,1\}^* \cdot 1$

Then $\varphi(w) = \psi(w)$ iff $\overline{0.\varphi(w)}^2 = \overline{0.\psi(w)}^2$

iff $\overline{0.\varphi(w)}^2 + (1 - \overline{0.\psi(w)}^2) = 1$

iff $\frac{1}{2}\overline{0.\varphi(w)}^2 + \frac{1}{2}(1 - \overline{0.\psi(w)}^2) = \frac{1}{2}$

Consider



$$[\![\mathcal{A}]\!](aw) = \frac{1}{2}\overline{0.\varphi(w)}^2 + \frac{1}{2}\left(1 - \overline{0.\psi(w)}^2\right)$$

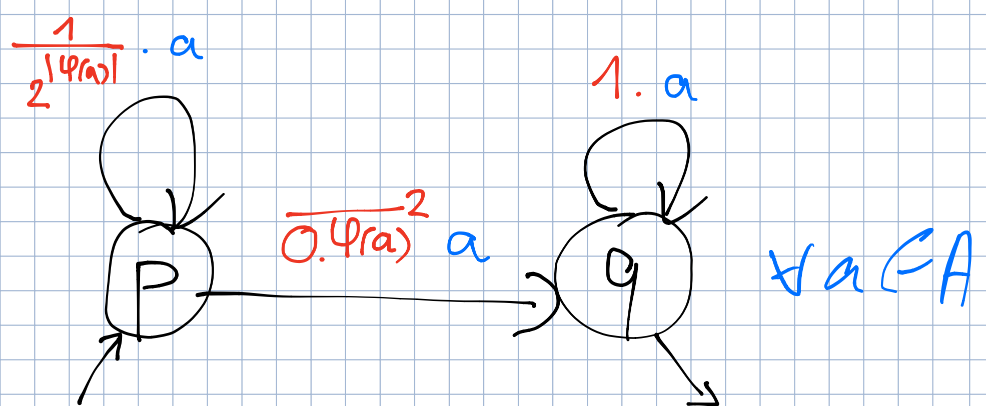
Hence $\exists w \in A^*$ $[\![\mathcal{A}]\!](aw) = \frac{1}{2}$ iff PCP has a solution

Lemma 1: Let $\varphi : A^* \longrightarrow \{0,1\}$ be a morphism

We build $\mathcal{A}_\varphi$ which computes $\overline{0.\varphi(w)}^2$ $\forall w \in A^+$

Proof:

$\mathcal{A}_\varphi$



Example: $\varphi(a) = 101$ $\quad \varphi(b) = 001$

Compute $[\![\mathcal{A}_\varphi]\!](ab) = \overline{0.\varphi(ab)}^2$

2 paths:

$$p \xrightarrow[0.101]{a} q \xrightarrow[1]{b} q \qquad 0.101 \times 1 = 0.101$$

$$p \xrightarrow[\frac{1}{2^3}]{a} p \xrightarrow[0.001]{b} q \qquad \frac{1}{2^3} \times 0.001 = 0.000\,001$$

$$\text{Sum} = 0.101\,001 = \overline{0.\varphi(ab)}^2$$

Weight of an accepting path over $w = a_1 a_2 \cdots a_n$

$$p \xrightarrow{a_1} p \xrightarrow{a_2} p \quad \cdots \quad p \xrightarrow{a_i} q \cdots \quad q \xrightarrow{a_n} q$$

$$\frac{1}{2^{|\varphi(a_1)|}} \times \cdots \times \frac{1}{2^{|\varphi(a_{i-1})|}} \times \overline{0.\varphi(a_i)}^2 \times 1 \times \cdots \times 1$$

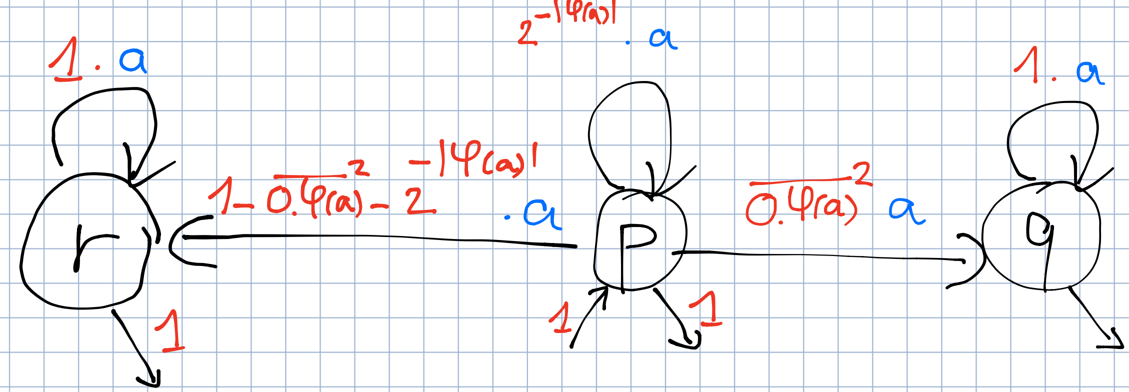
$$= \frac{1}{2^{|\varphi(a_1 \cdots a_{i-1})|}} \times \overline{0.\varphi(a_i)}^2 = \overline{0.\underbrace{0 \cdots 0}_{|\varphi(a_1 \cdots a_{i-1})|}\varphi(a_i)}^2$$

$$[\![\mathcal{A}]\!](w) = \overline{0.\varphi(a_1)}^2 + \overline{0.0 \cdots 0\varphi(a_2)}^2 + \overline{0.0 \cdots\cdots 0\varphi(a_3)}^2 + \cdots + \overline{0.0 \cdots\cdots\cdots\cdots 0\varphi(a_n)}^2 = \overline{0.\varphi(w)}^2 \quad \square$$

Lemma 2: Let $\varphi : A^* \longrightarrow \{0,1\}$ be a morphism

We build $\mathcal{A}_{1-\varphi}$ which computes $1-\overline{0.\varphi(w)}^2 \ \forall w \in A^*$

Proof:

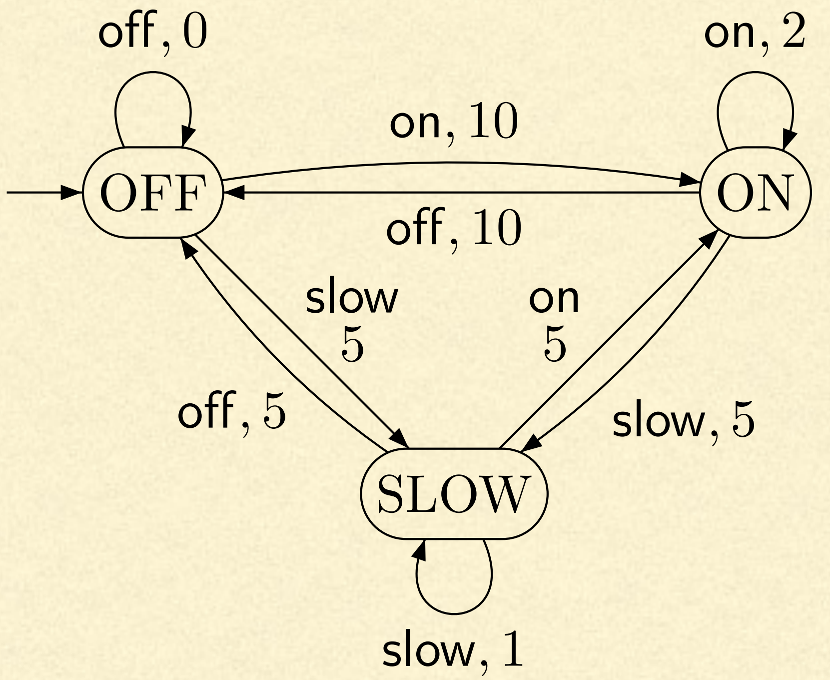


WEIGHTED AUTOMATA

1. Definitions, Examples, Various semantics
2. Boolean vs Quantitative languages
3. Some decision problems
4. Extensions I: Infinite words, Trees, Pictures, Graphs
5. Extensions II: Pebble Walking Automata

INFINITE WORDS

- Energy consumption: limit average



- sup, inf $\quad \text{Val}(s_0 s_1 s_2 \cdots) = \sup_n s_n$
- limit sup, limit inf $\quad \text{Val}(s_0 s_1 s_2 \cdots) = \limsup_n s_n$
- limit average $\quad \text{Val}(s_0 s_1 s_2 \cdots) = \liminf_n \frac{s_0 + \cdots + s_{n-1}}{n}$
- discounted sum $\quad \text{Val}(s_0 s_1 s_2 \cdots) = \sum_n \lambda^n s_n$

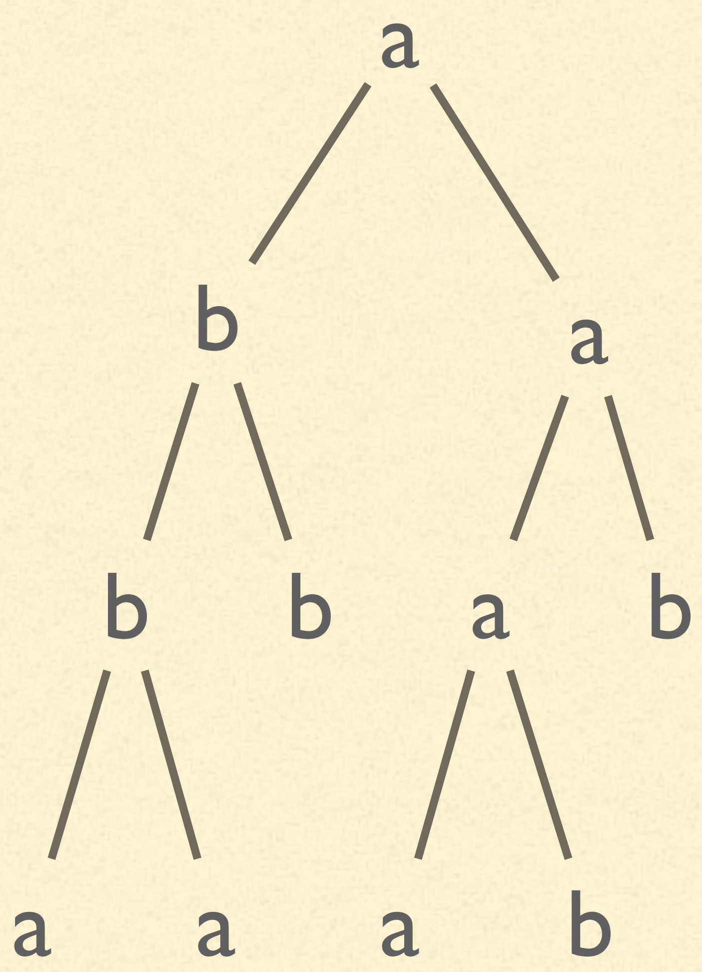
INFINITE WORDS

Chatterjee Doyen Henzinger 2010:

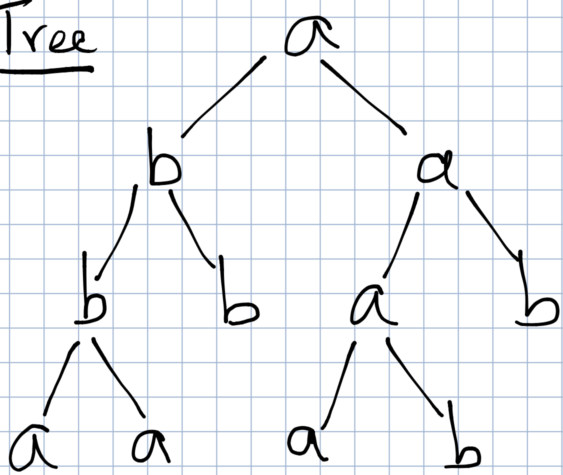
Study of the decision problems: Threshold, inclusion, equivalence

- sup, inf — $\mathsf{Val}(s_0 s_1 s_2 \cdots) = \sup_n s_n$
- limit sup, limit inf — $\mathsf{Val}(s_0 s_1 s_2 \cdots) = \limsup_n s_n$
- limit average — $\mathsf{Val}(s_0 s_1 s_2 \cdots) = \liminf_n \frac{s_0 + \cdots + s_{n-1}}{n}$
- discounted sum — $\mathsf{Val}(s_0 s_1 s_2 \cdots) = \sum_n \lambda^n s_n$

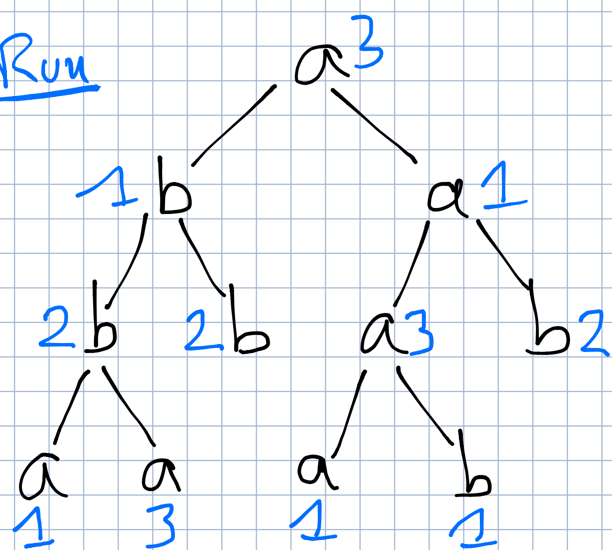
TREES



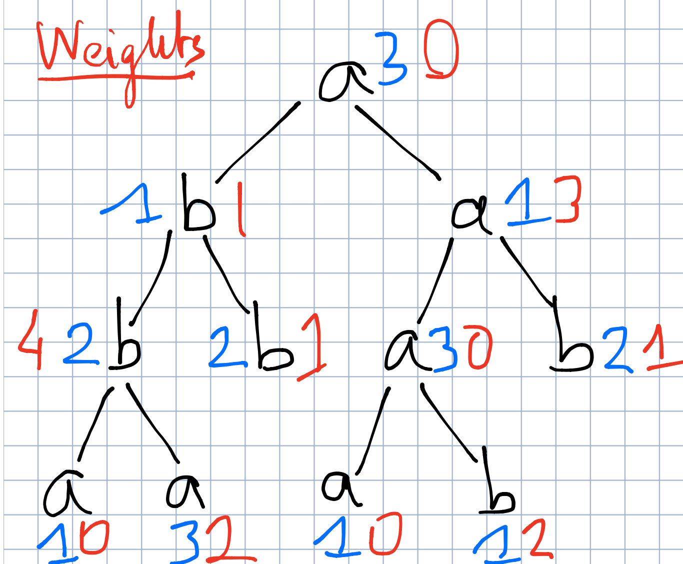
Tree



Run

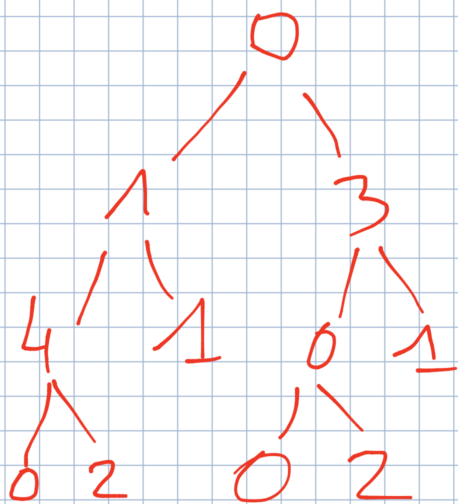


Weights



Transitions:	Weights
$\xrightarrow{a} 1$	0
$\xrightarrow{a} 3$	2
$\xrightarrow{b} 1$	2
$\xrightarrow{b} 2$	1
$1,1 \xrightarrow{a} 3$	0
$3,2 \xrightarrow{a} 1$	3
$1,3 \xrightarrow{b} 2$	4
$2,2 \xrightarrow{b} 1$	1

We obtain a <u>tree</u> of weights



Then we evaluate

ex_Sum 14

– Product 0

of nonzero 48

– Max over branches
Sum on the branch
7

WEIGHTED AUTOMATA

1. Definitions, Examples, Various semantics
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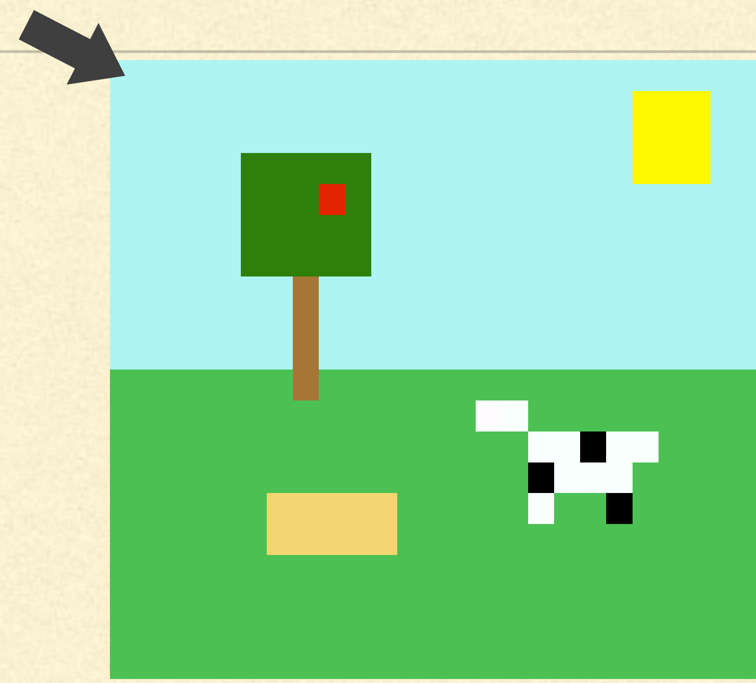
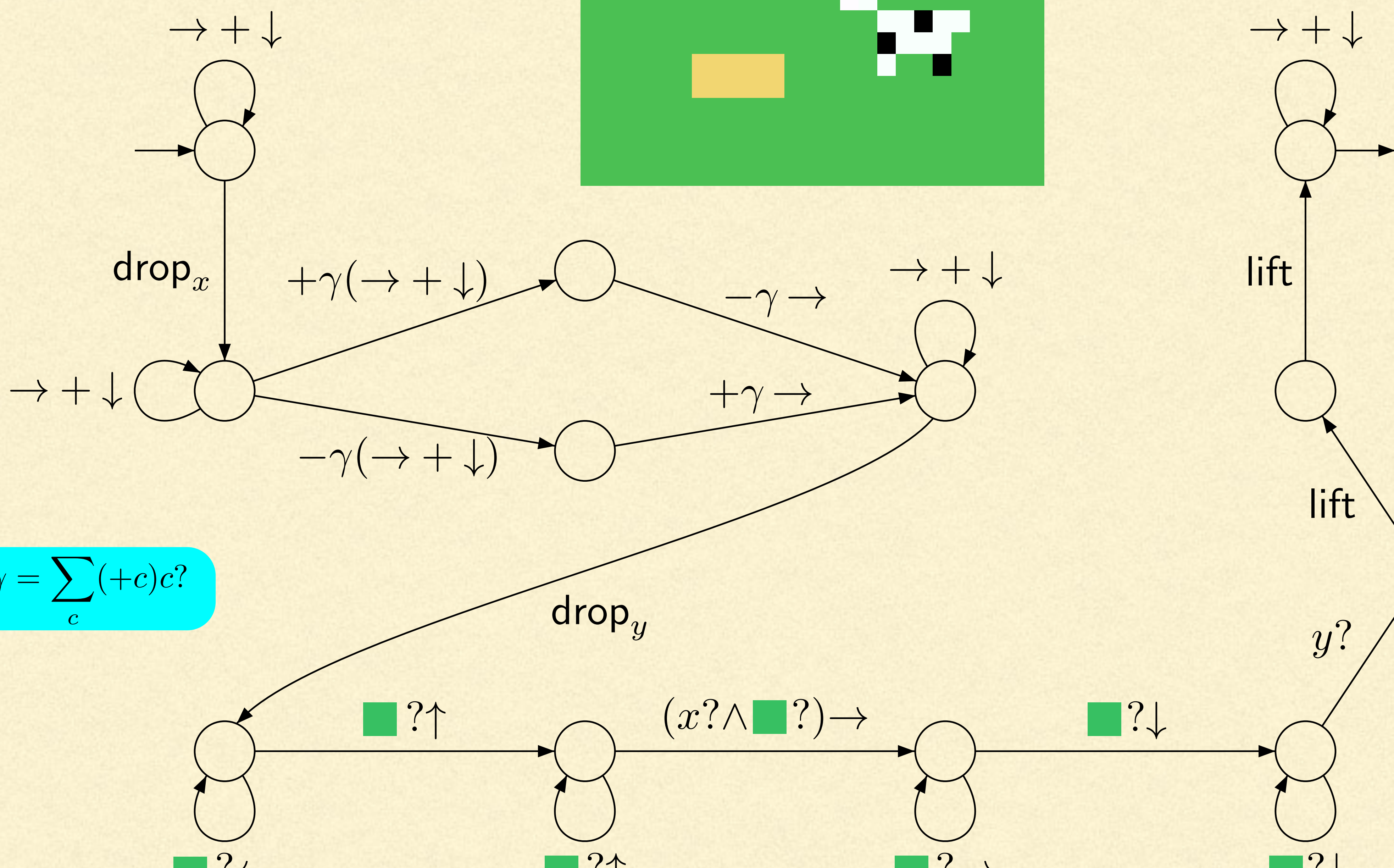
PEBBLE WALKING AUTOMATA

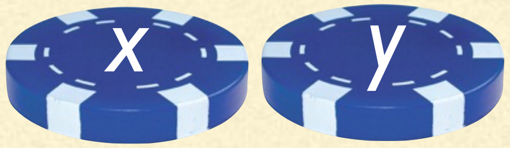


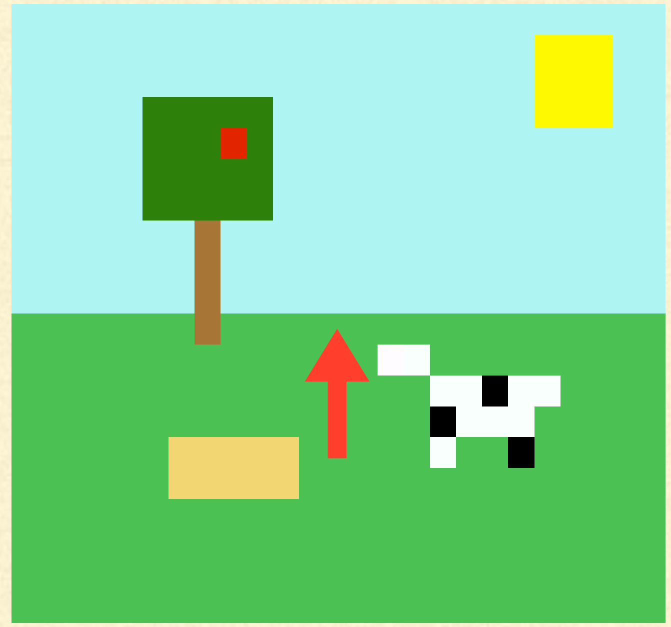
$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$



$\gamma = \sum_c (+c)c?$

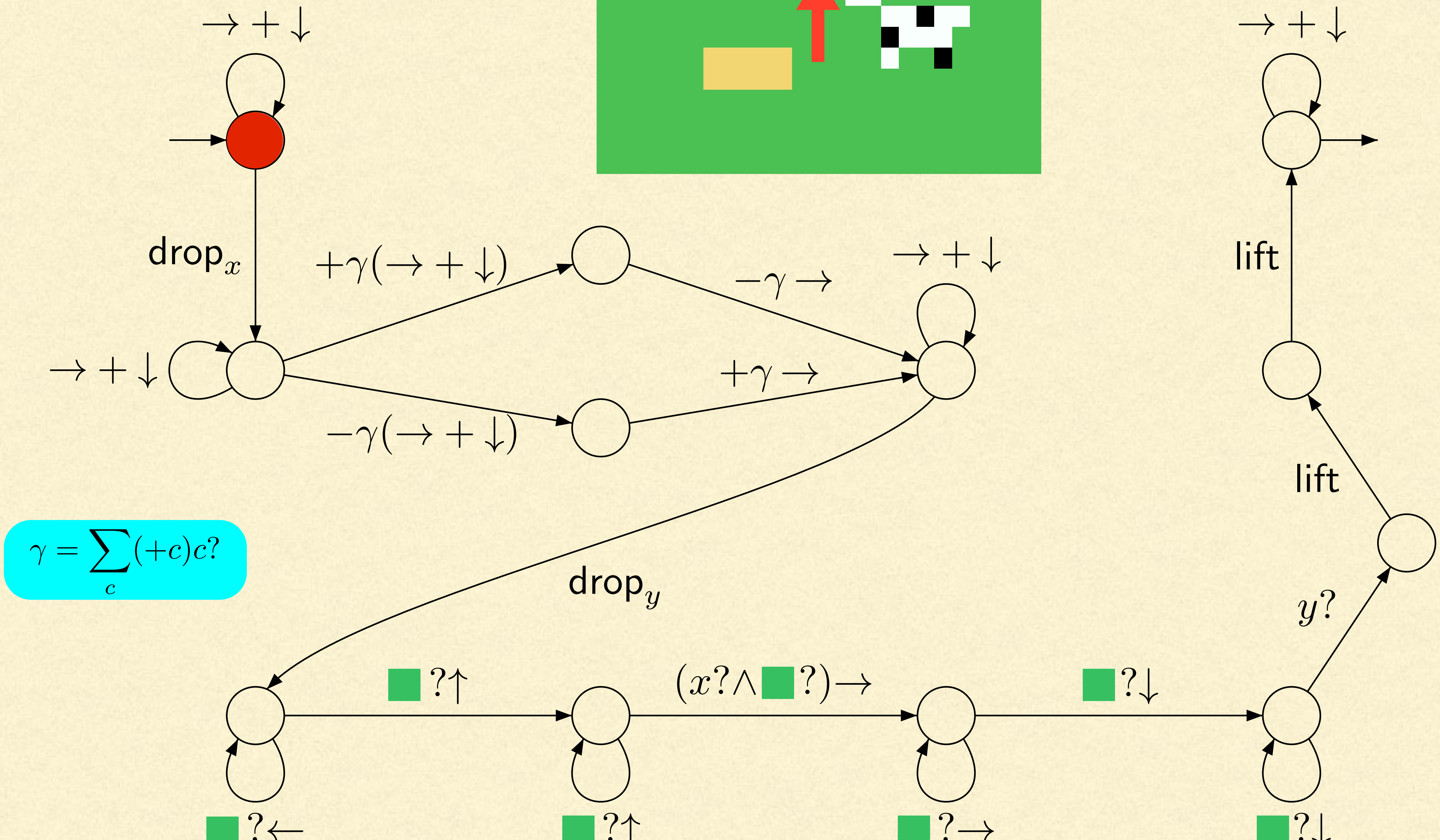
PEBBLE WALKING AUTOMATA



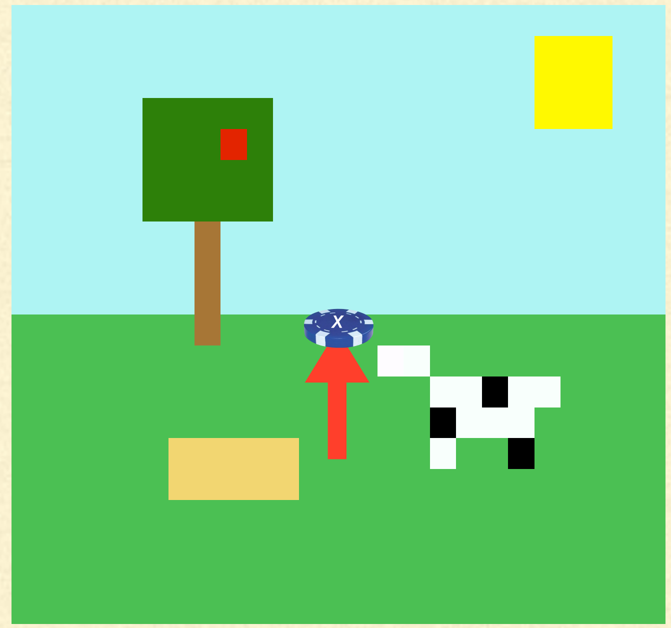


$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 ,

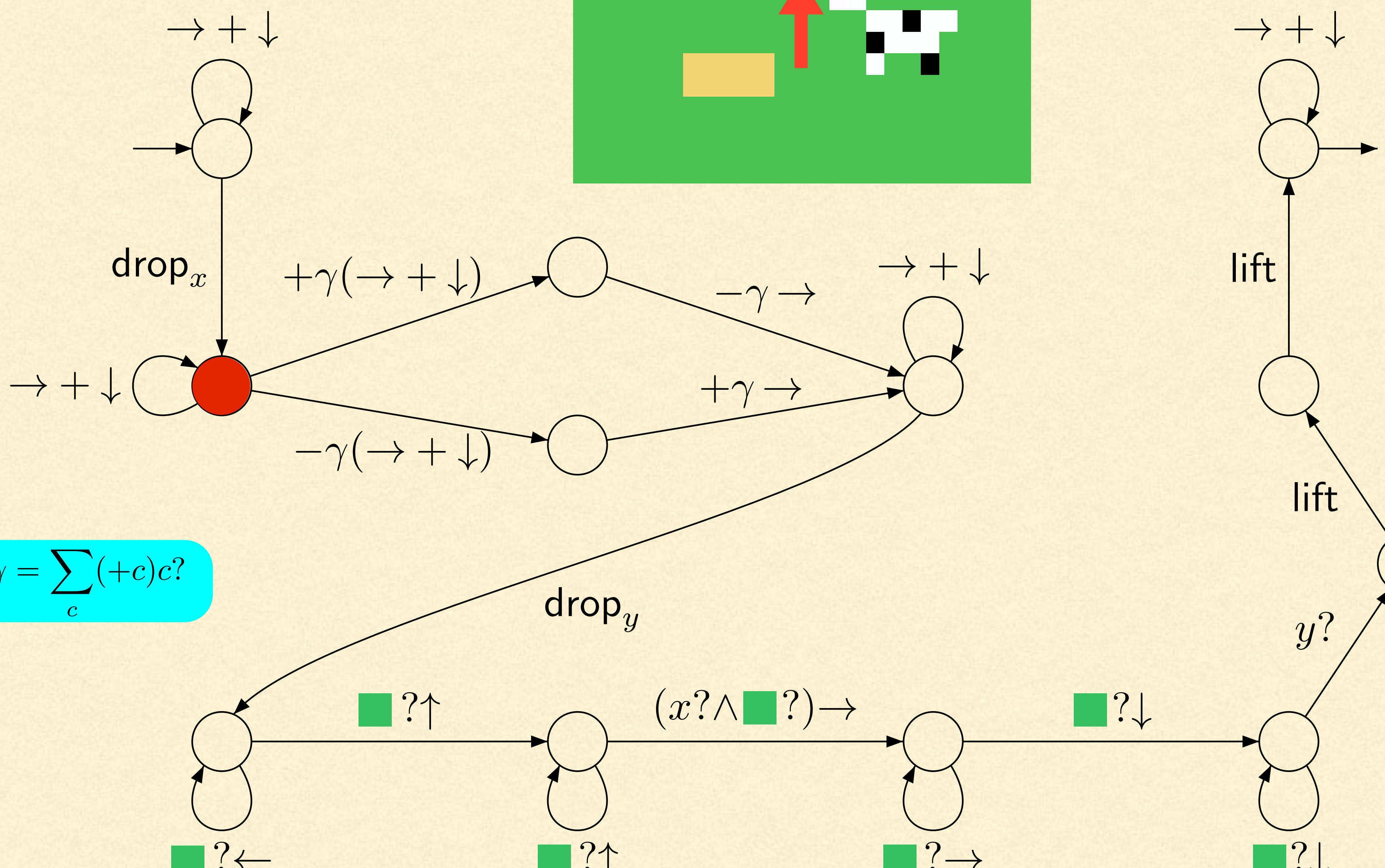


PEBBLE WALKING AUTOMATA



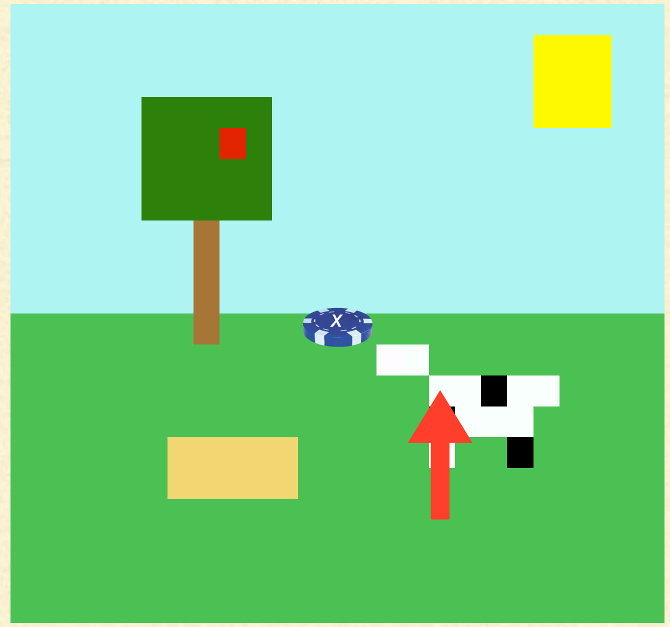
$$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$$

0 , … , 0 ,



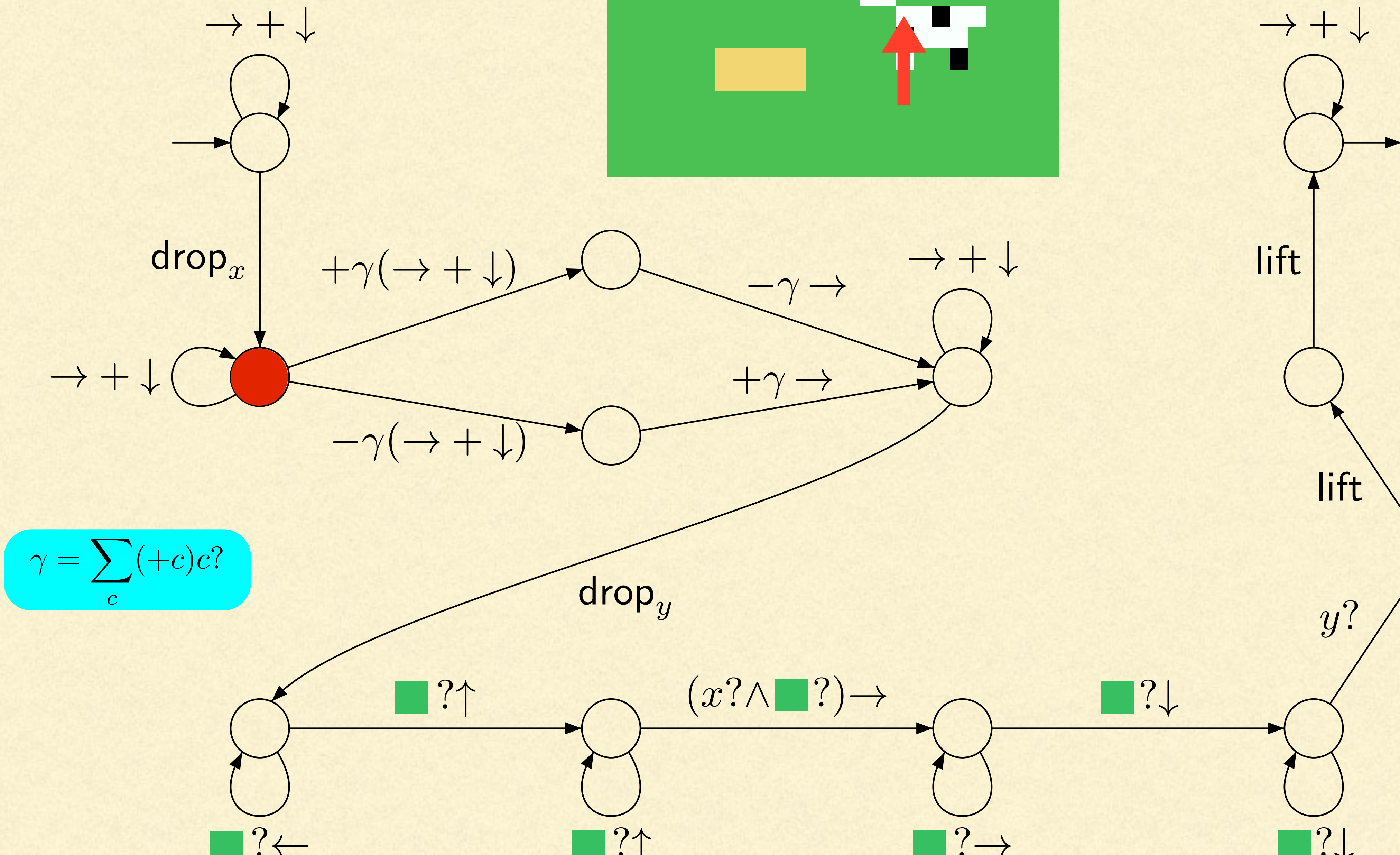
$$\gamma = \sum_c (+c)c?$$

PEBBLE WALKING AUTOMATA

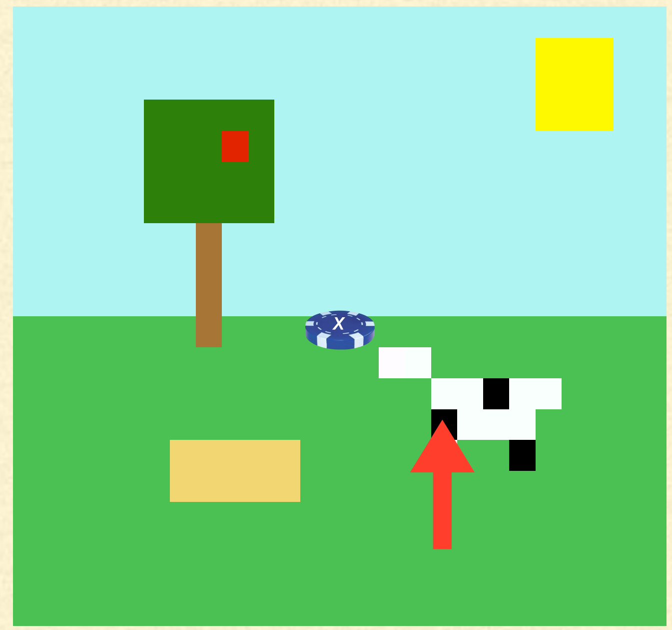


$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 ,

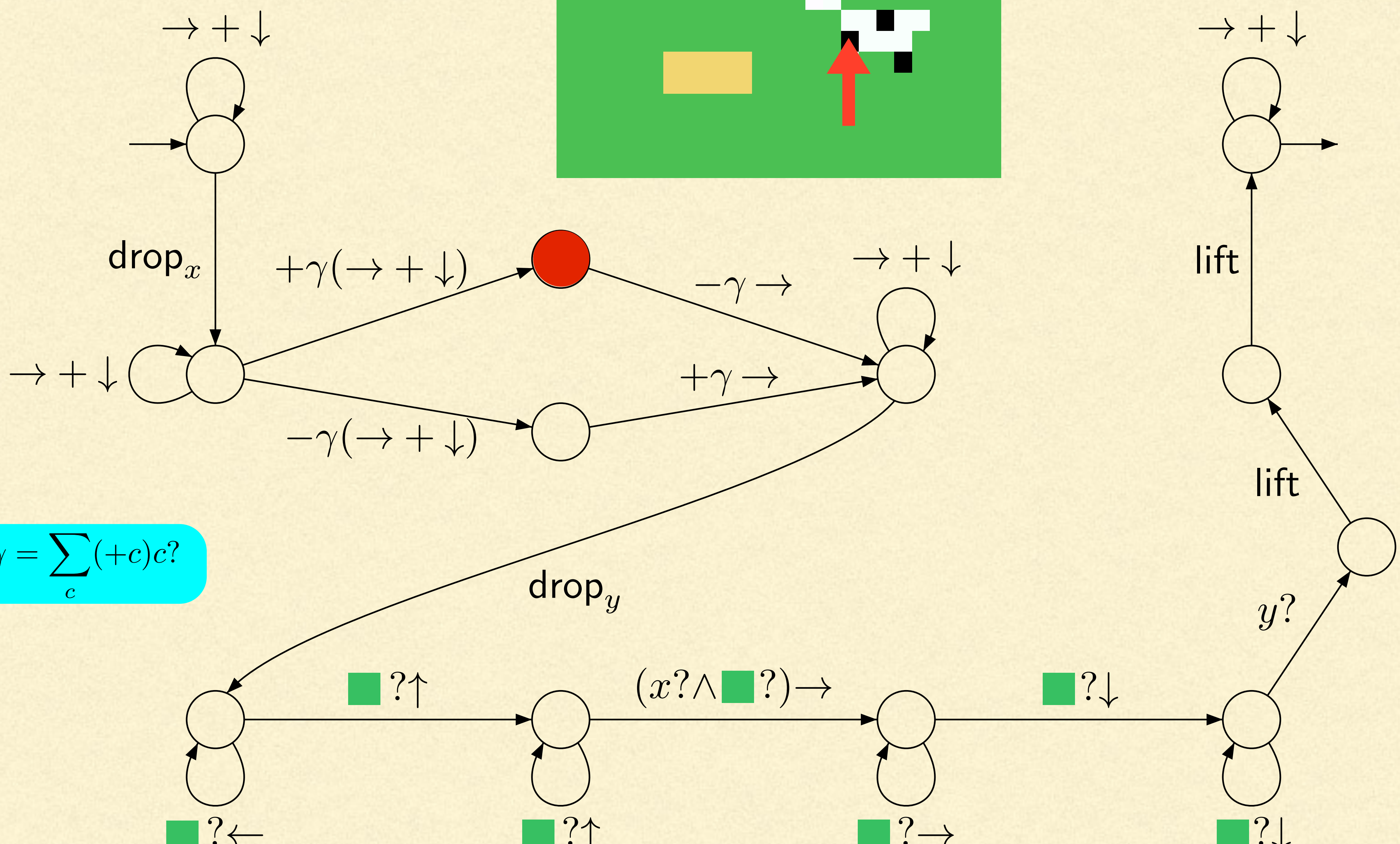


PEBBLE WALKING AUTOMATA



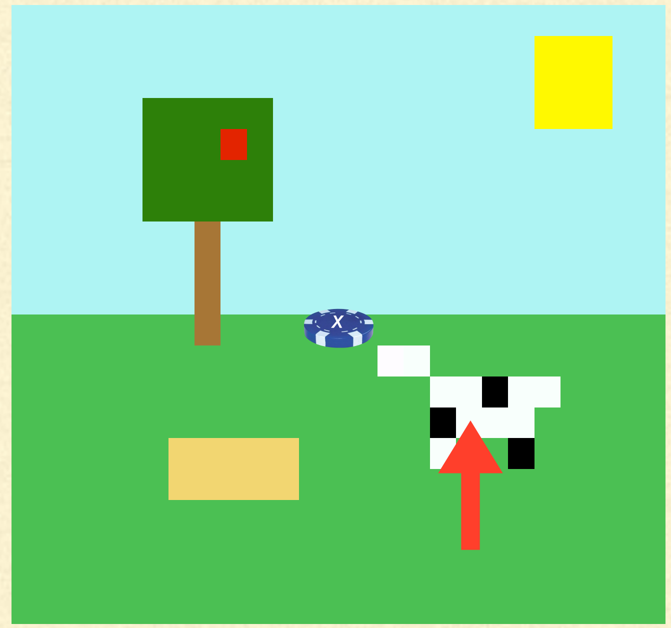
$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , +255 ,



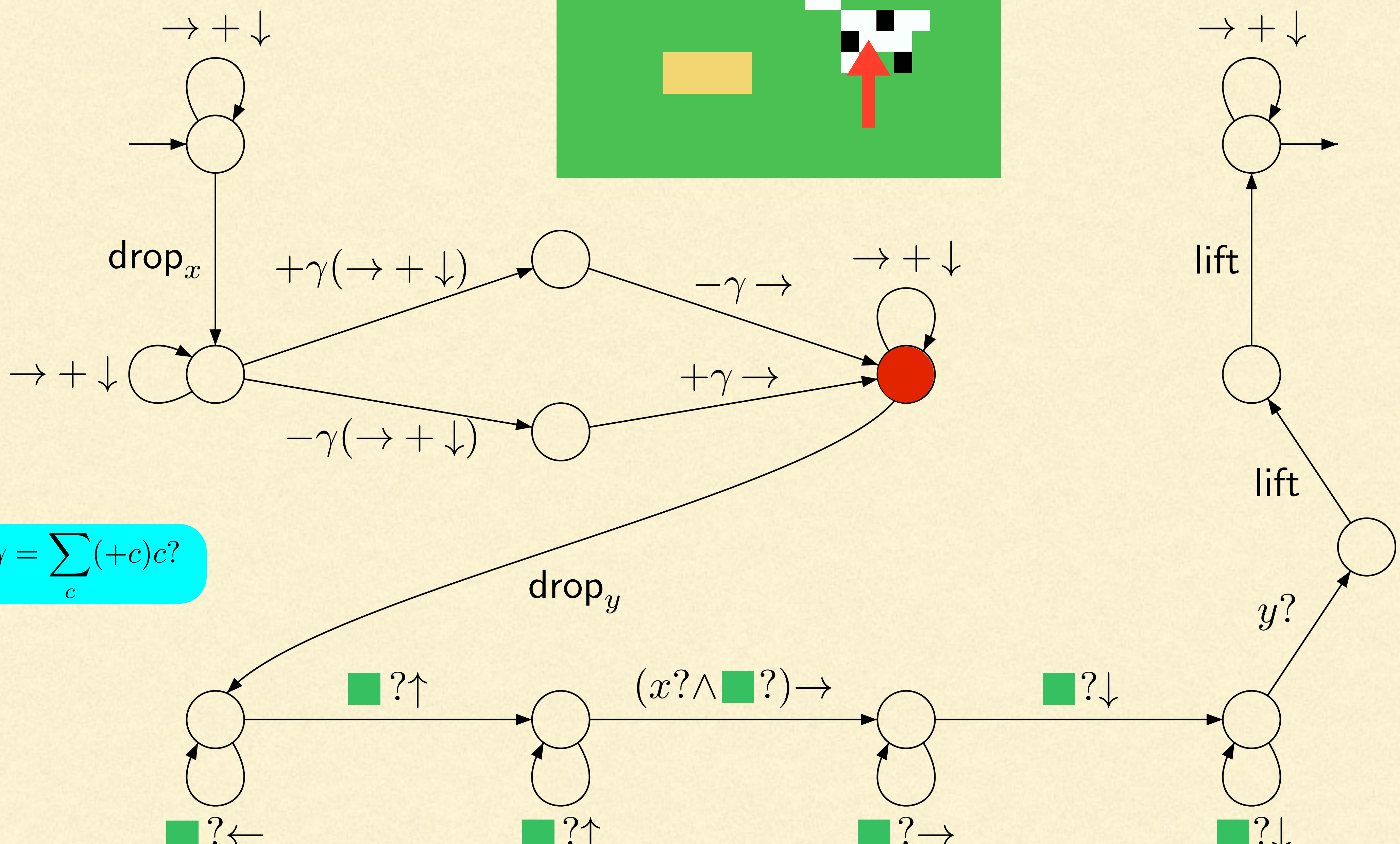
$\gamma = \sum_c (+c)c?$

PEBBLE WALKING AUTOMATA



$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , +255 , -0



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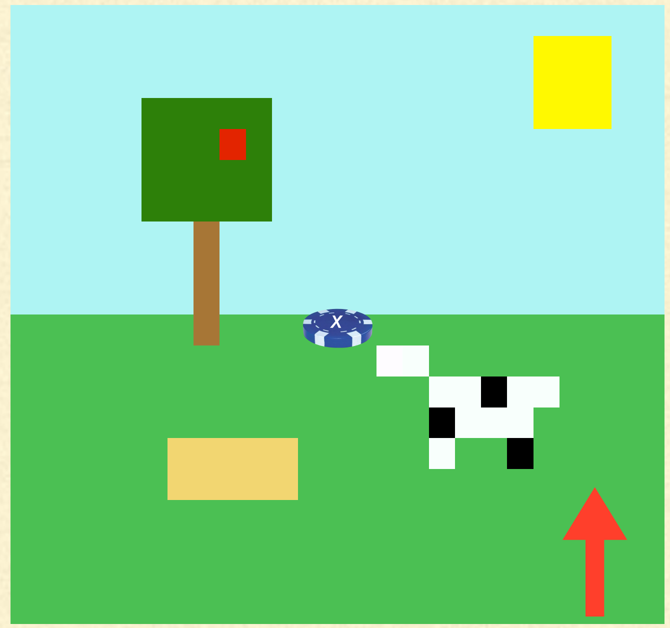
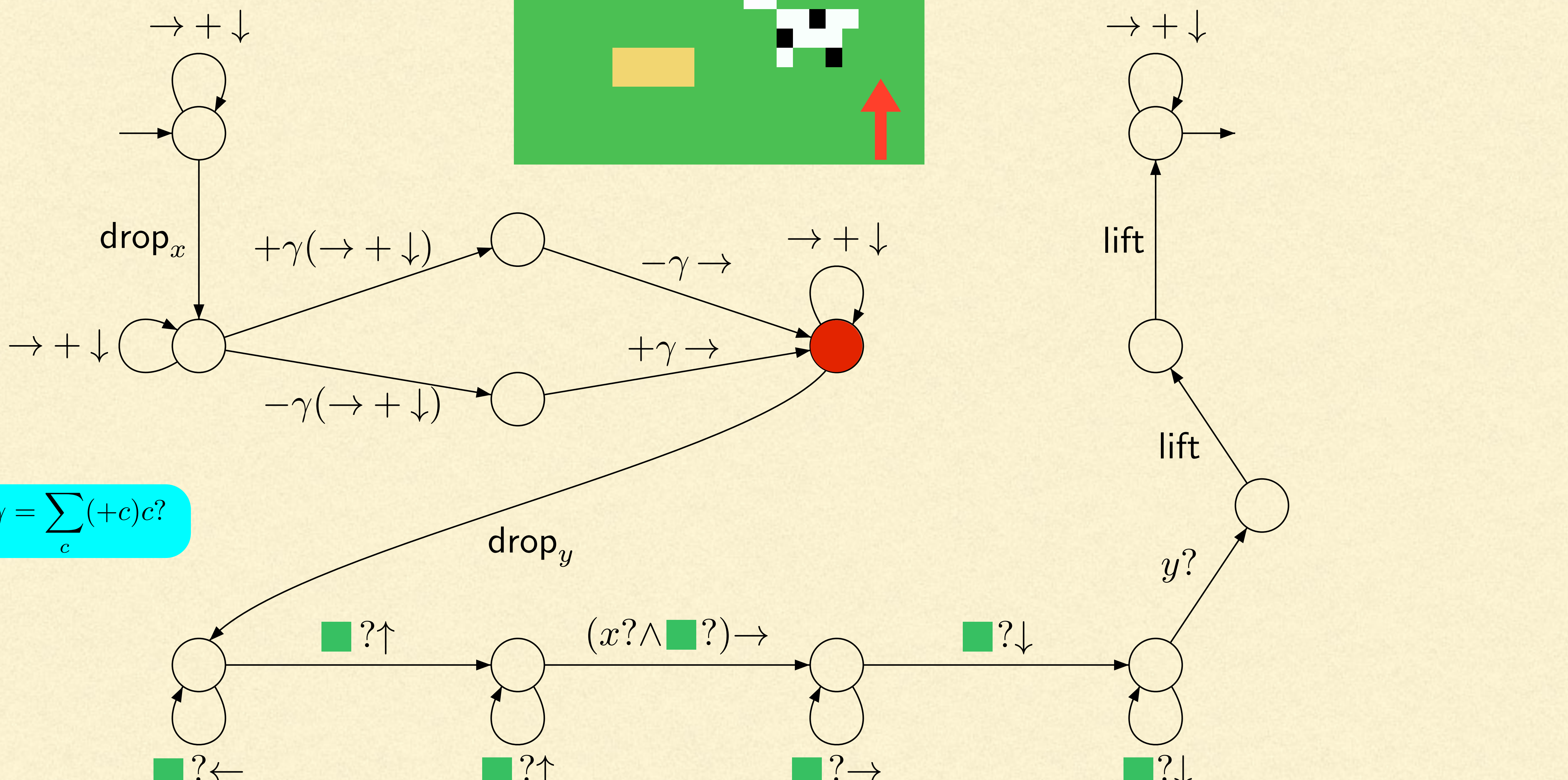


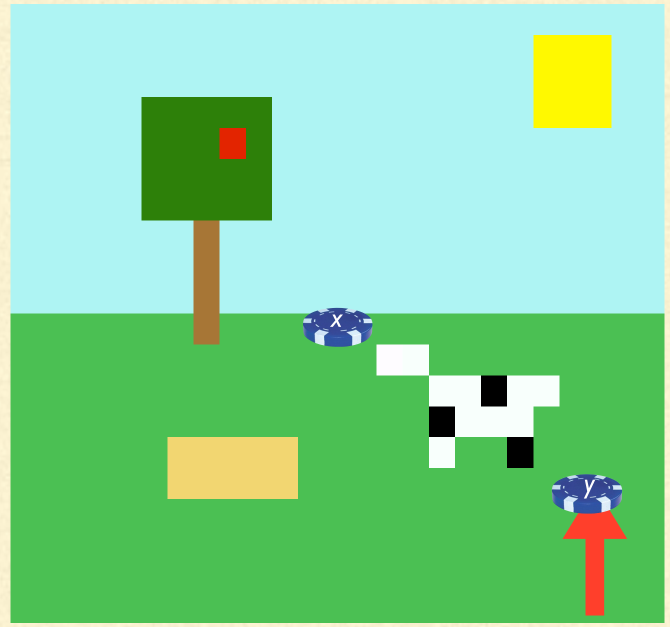
$$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$$

0 , … , 0 , +255 , -0



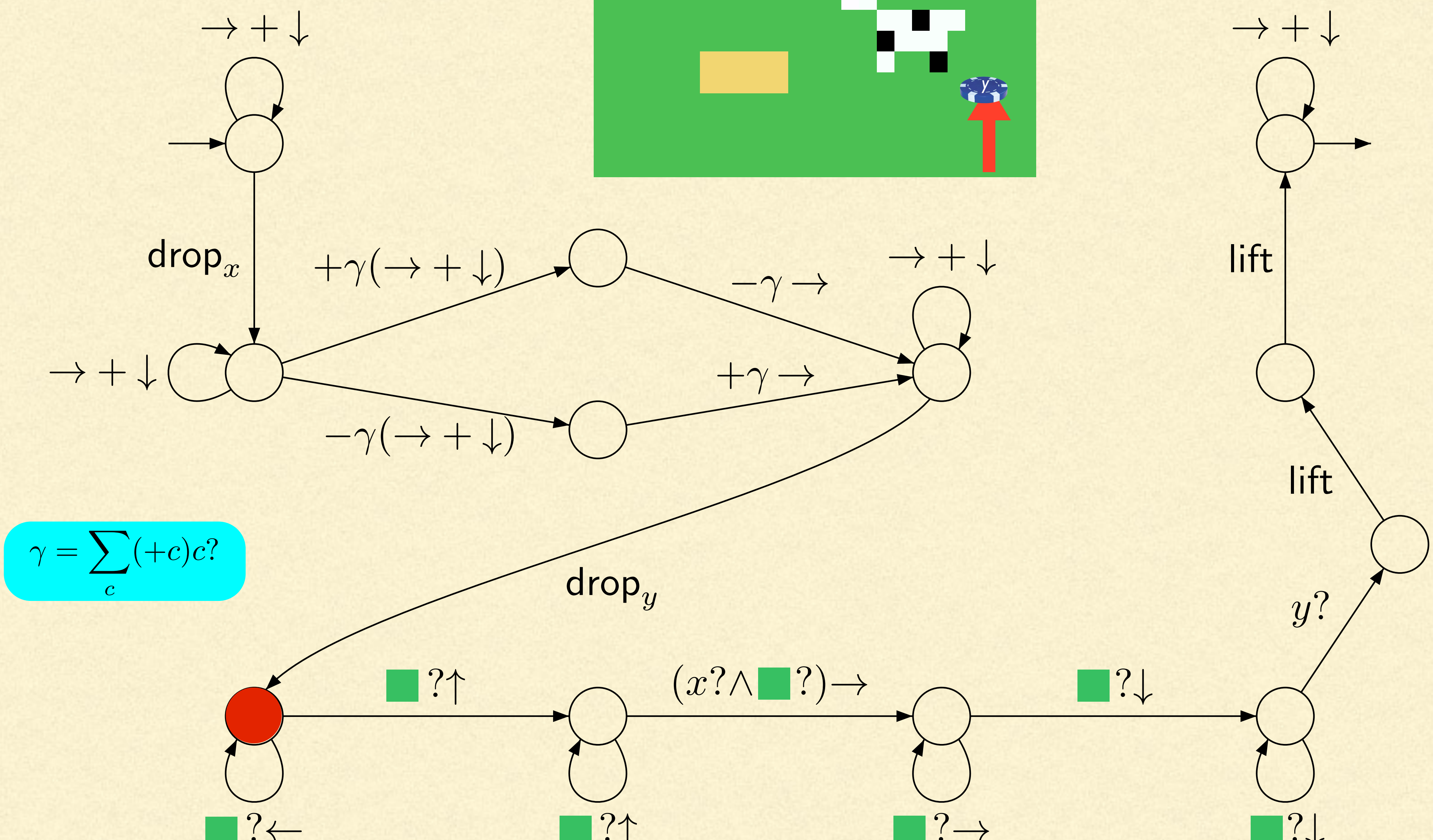
$$\gamma = \sum_c (+c)c?$$

PEBBLE WALKING AUTOMATA

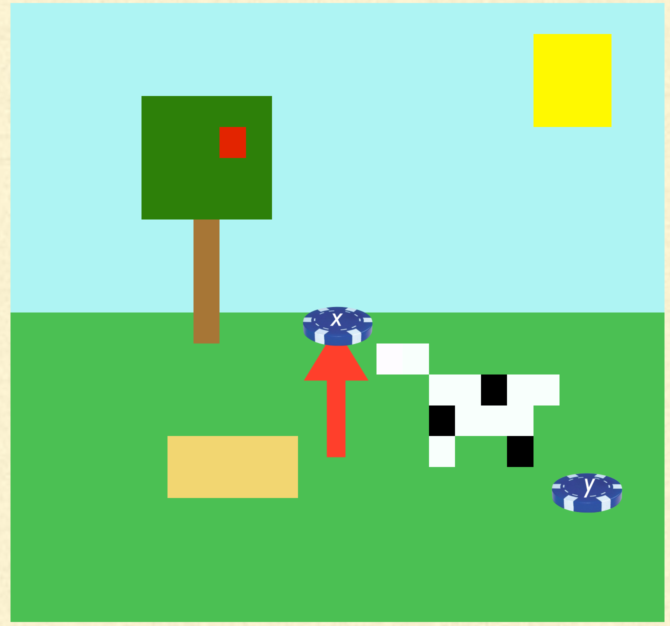


$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , +255 , -0

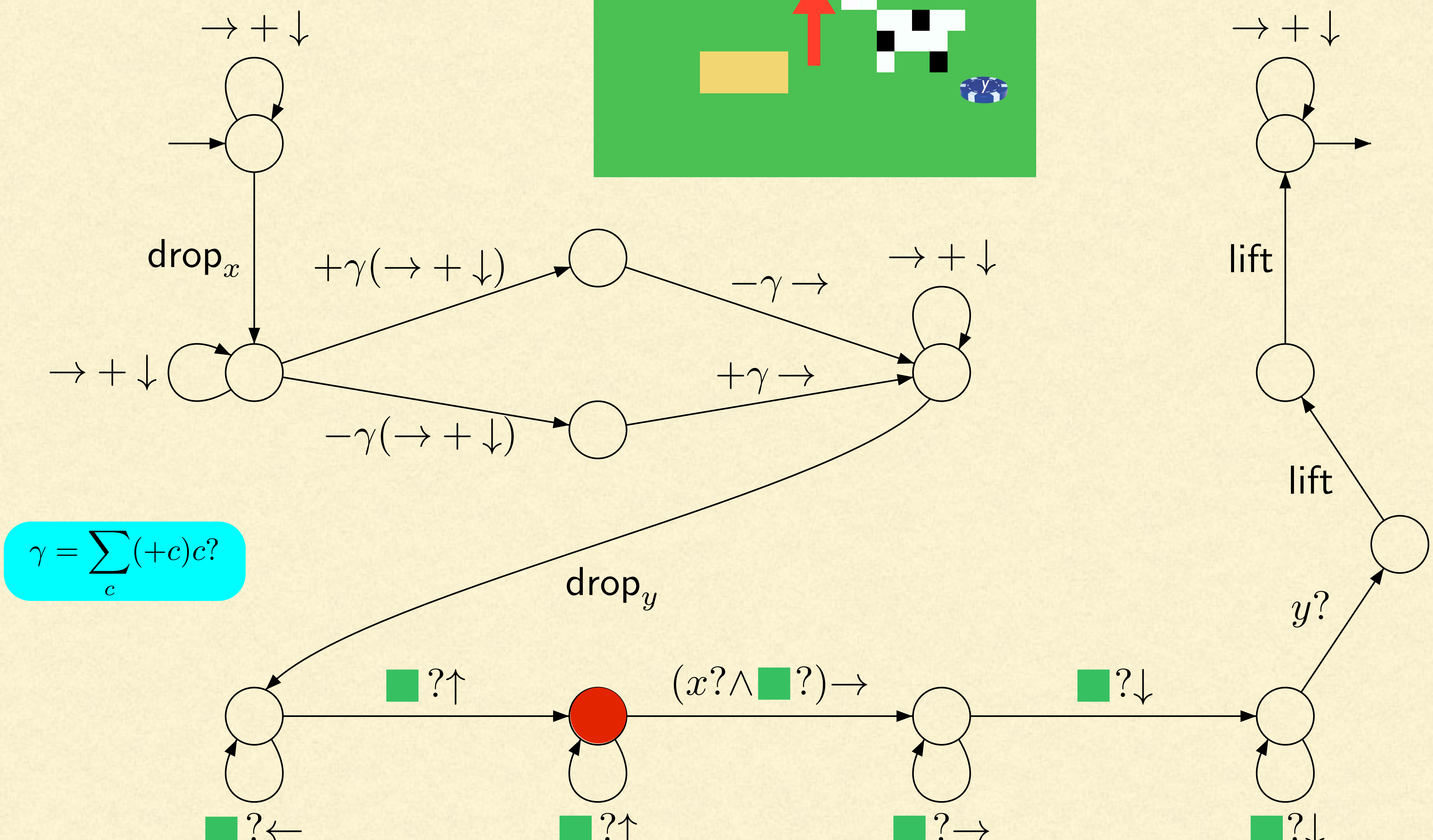


PEBBLE WALKING AUTOMATA

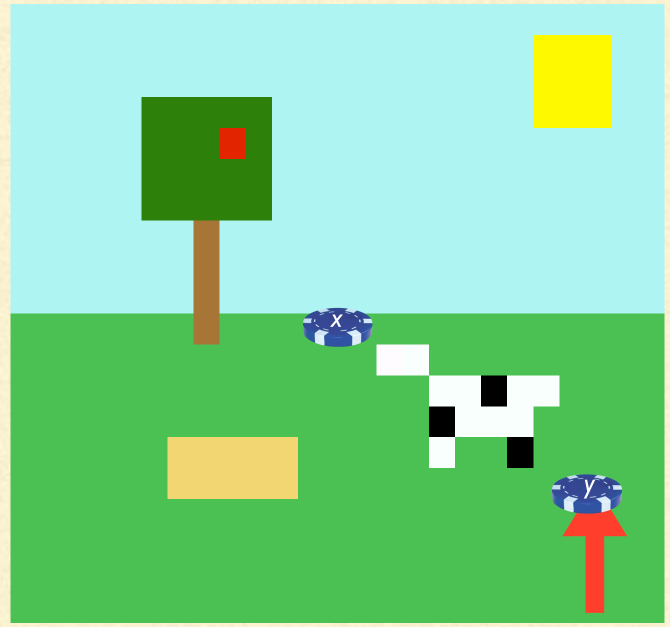


$$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$$

0 , … , 0 , **+255 , -0** , 0 , … , 0

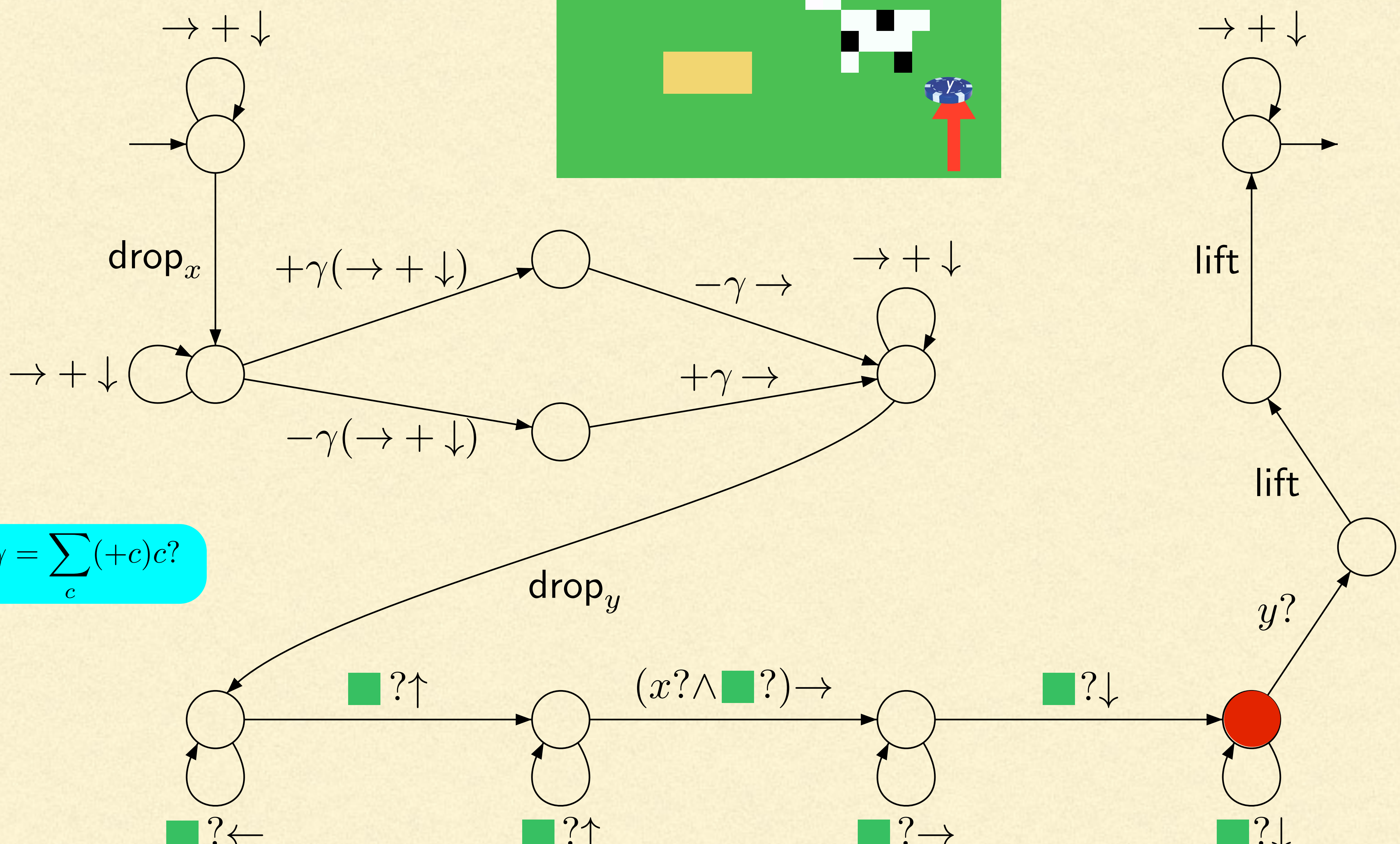


PEBBLE WALKING AUTOMATA



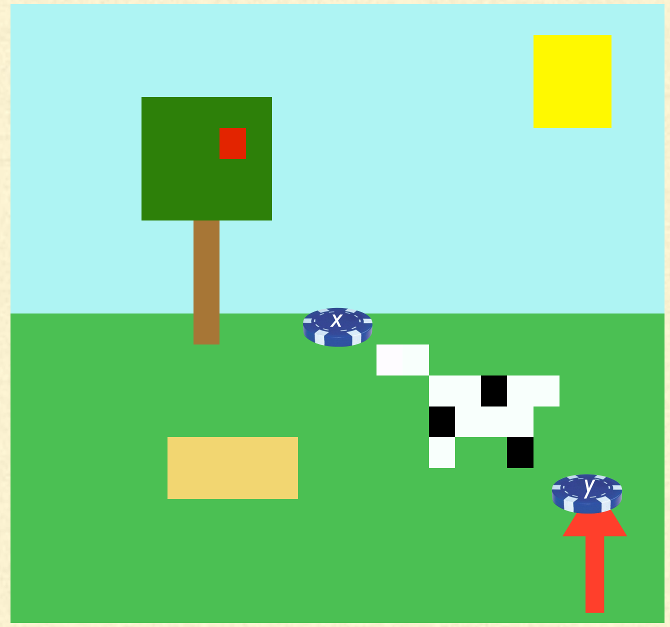
$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , +255 , -0 , 0 , … , 0



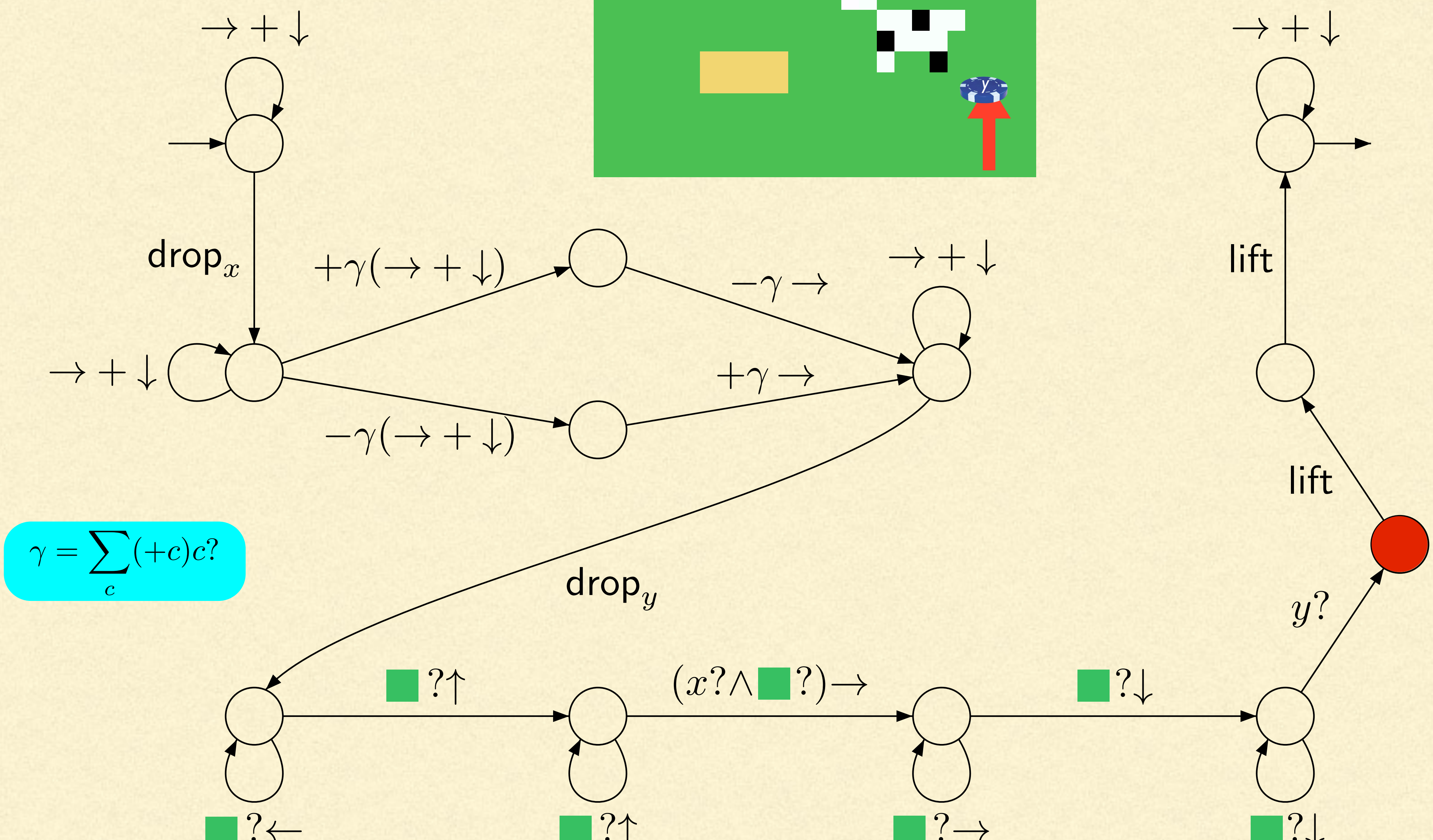
$\gamma = \sum_c (+c)c?$

PEBBLE WALKING AUTOMATA

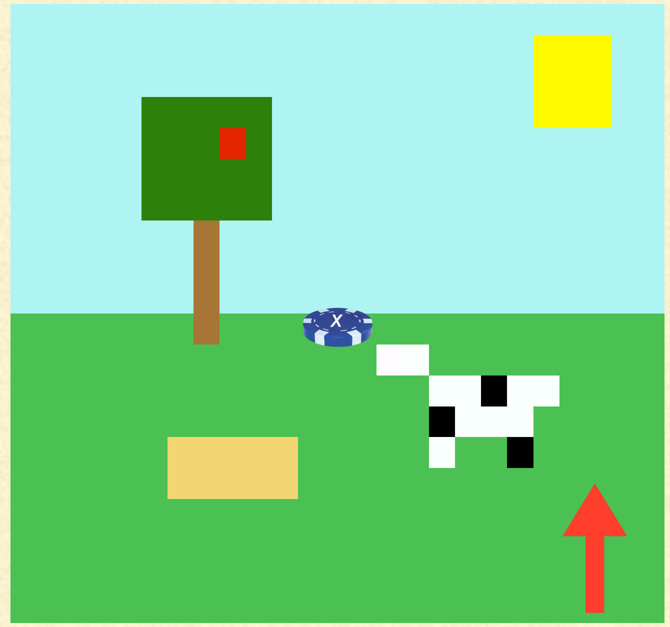


$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , **+255 , -0** , 0 , … , 0

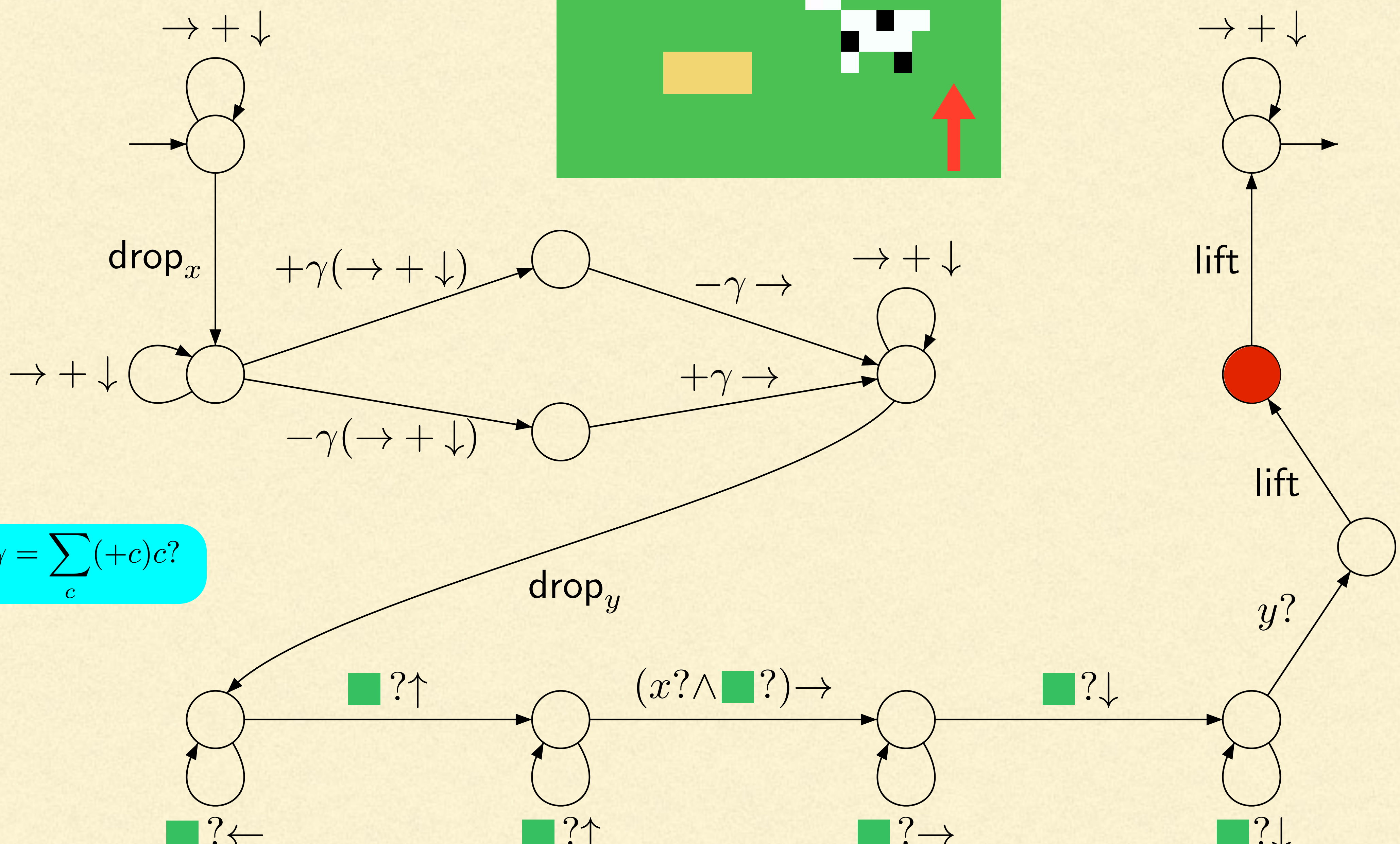


PEBBLE WALKING AUTOMATA



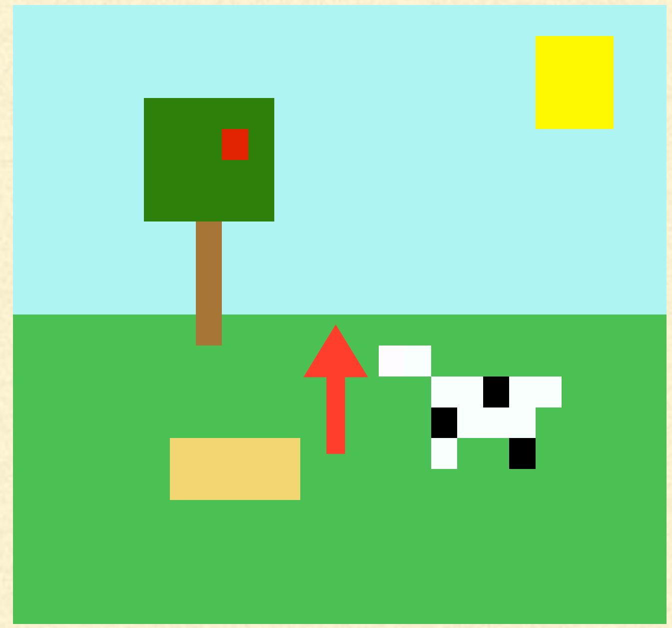
$$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$$

0 , … , 0 , **+255 , -0** , 0 , … , 0



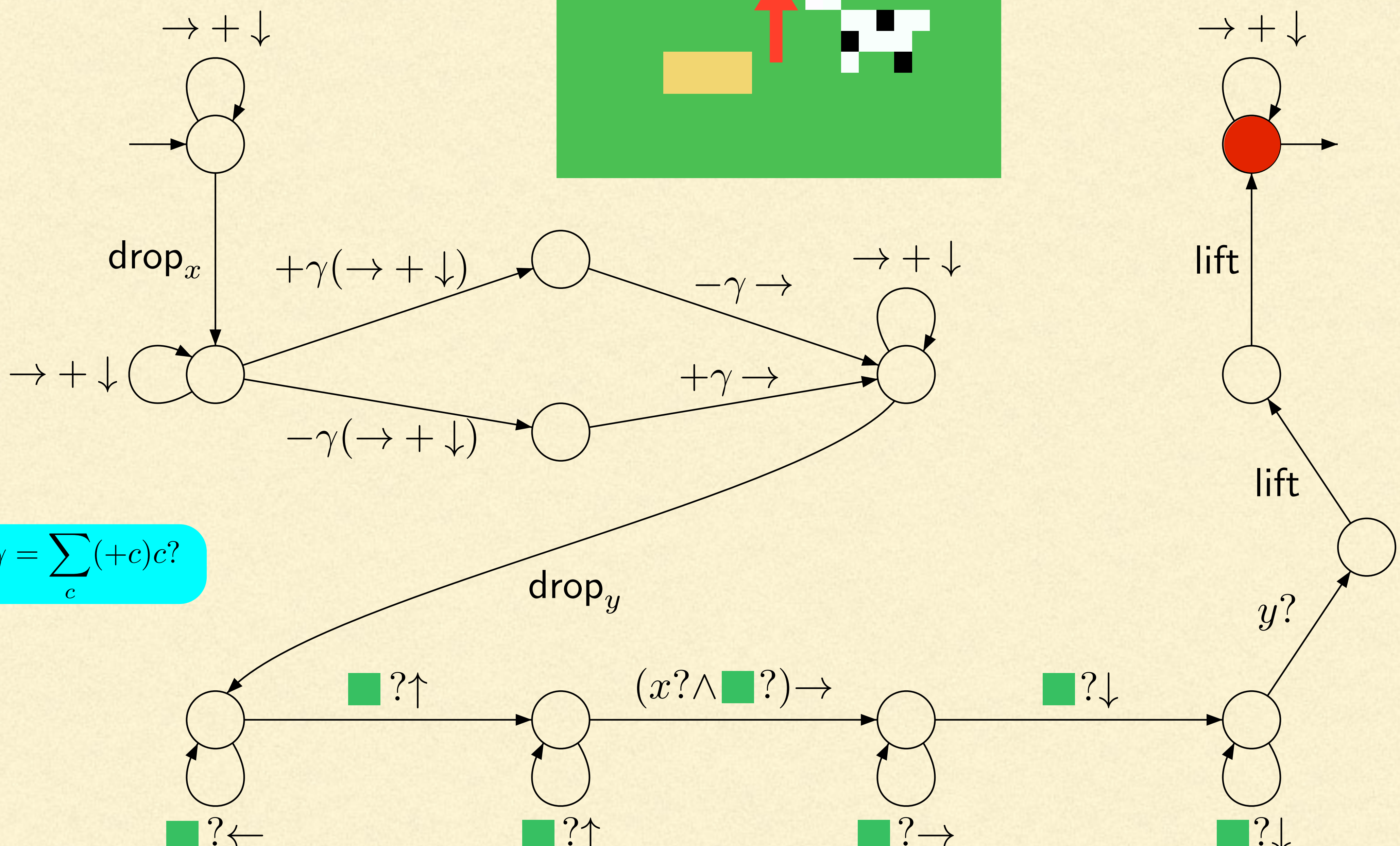
$$\gamma = \sum_c (+c)c?$$

PEBBLE WALKING AUTOMATA



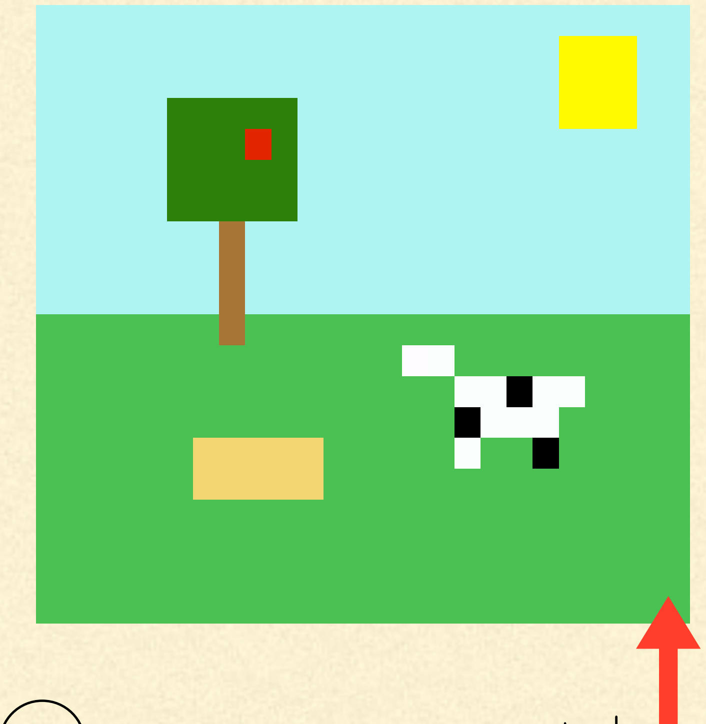
$$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$$

0 , … , 0 , **+255 , -0** , 0 , … , 0



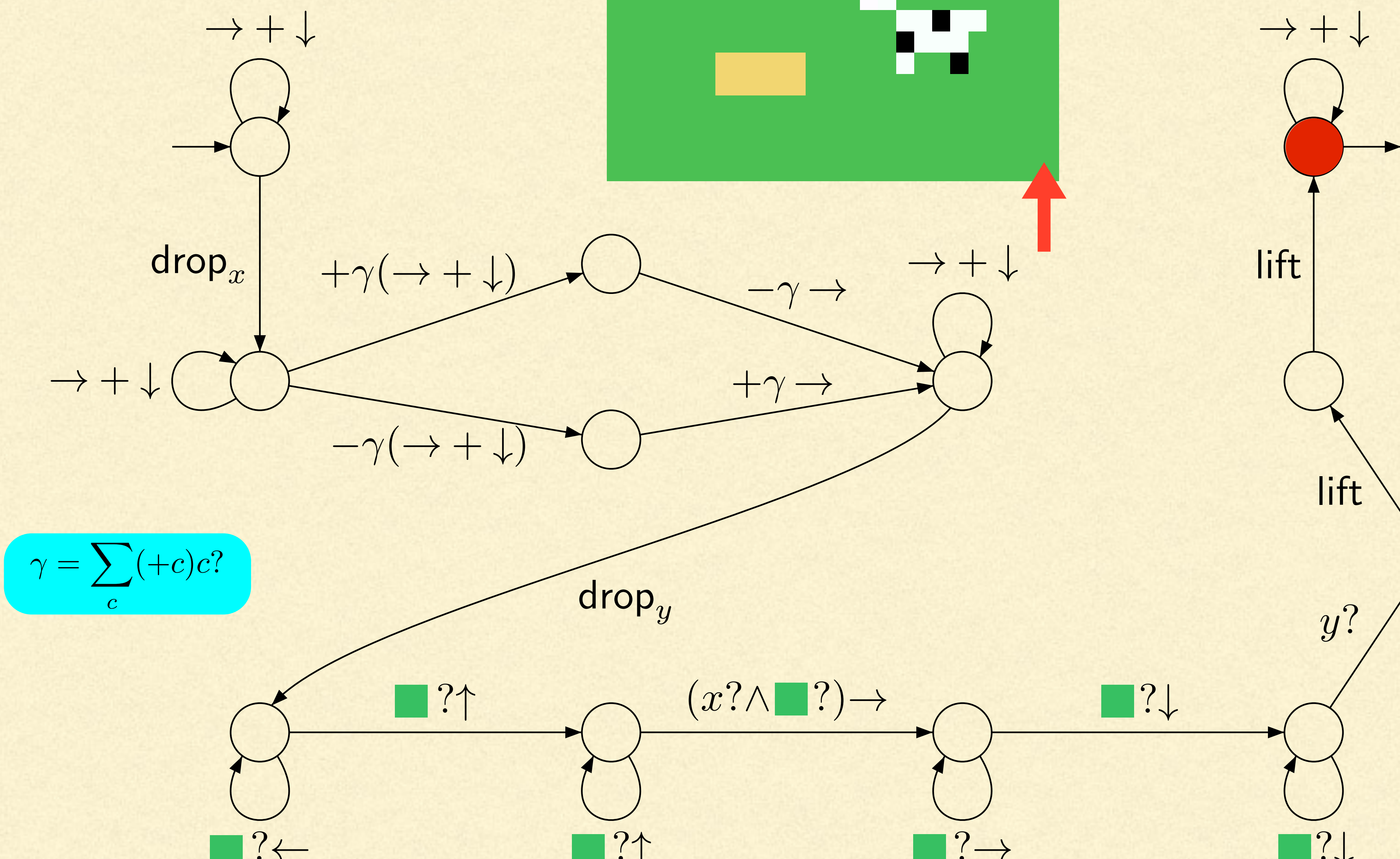
$$\gamma = \sum_c (+c) c?$$

PEBBLE WALKING AUTOMATA

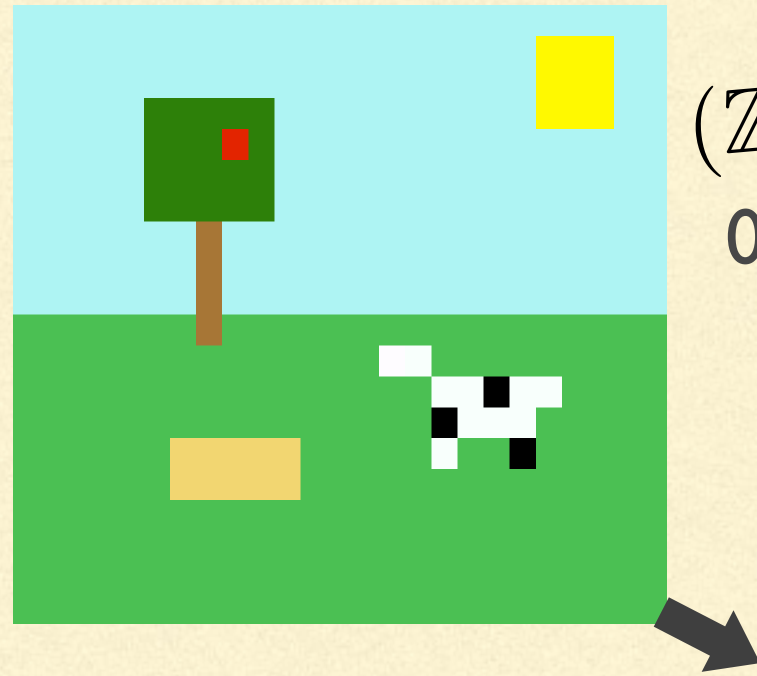


$(\mathbb{Z}\cup\{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , +255 , -0 , 0 , … , 0

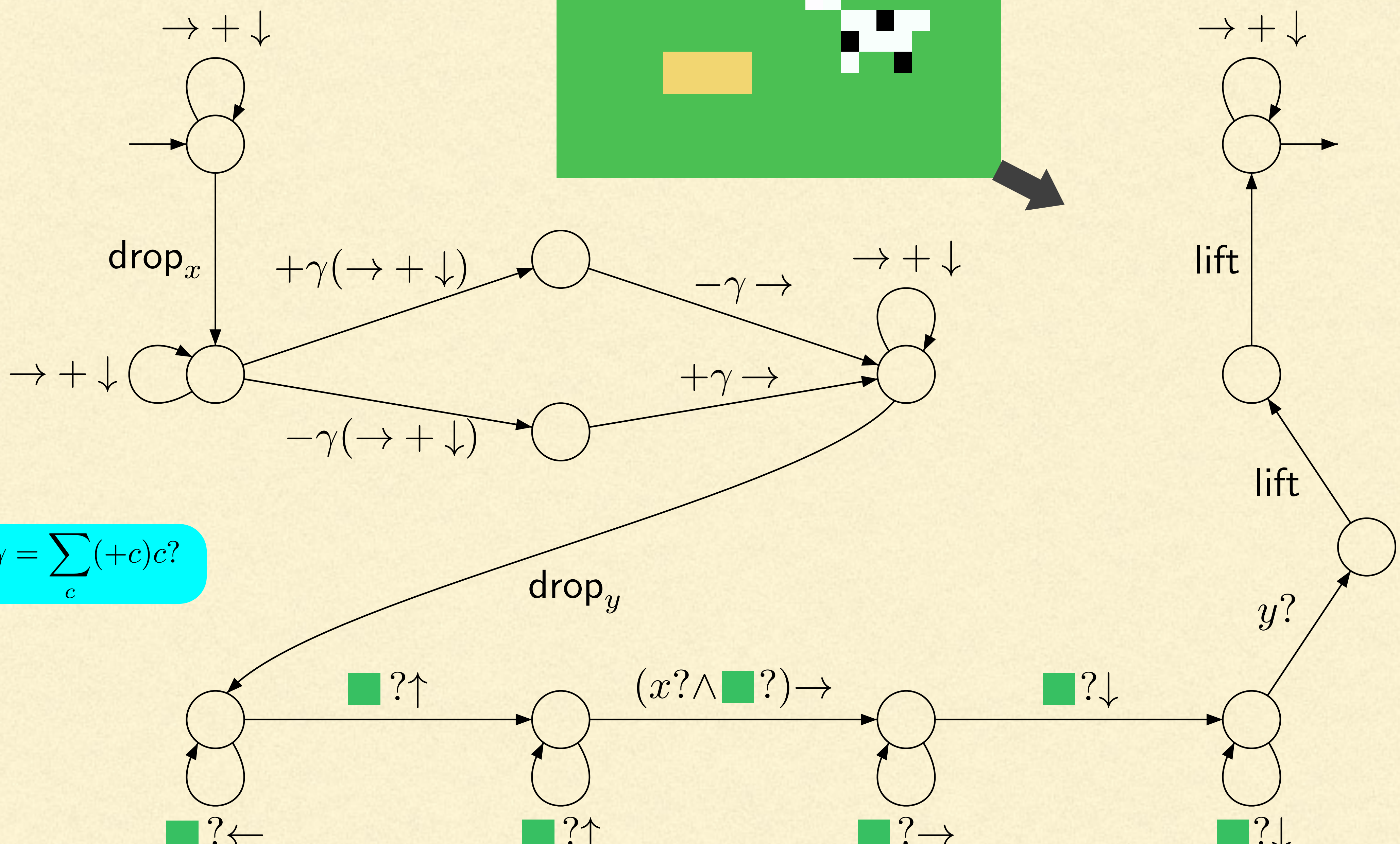


PEBBLE WALKING AUTOMATA



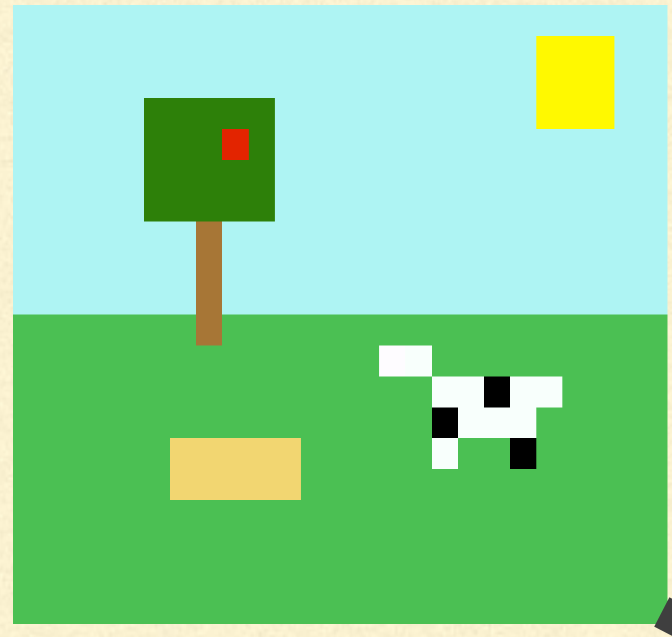
$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , **+255 , -0** , 0 , … , 0



$\gamma = \sum_c (+c)c?$

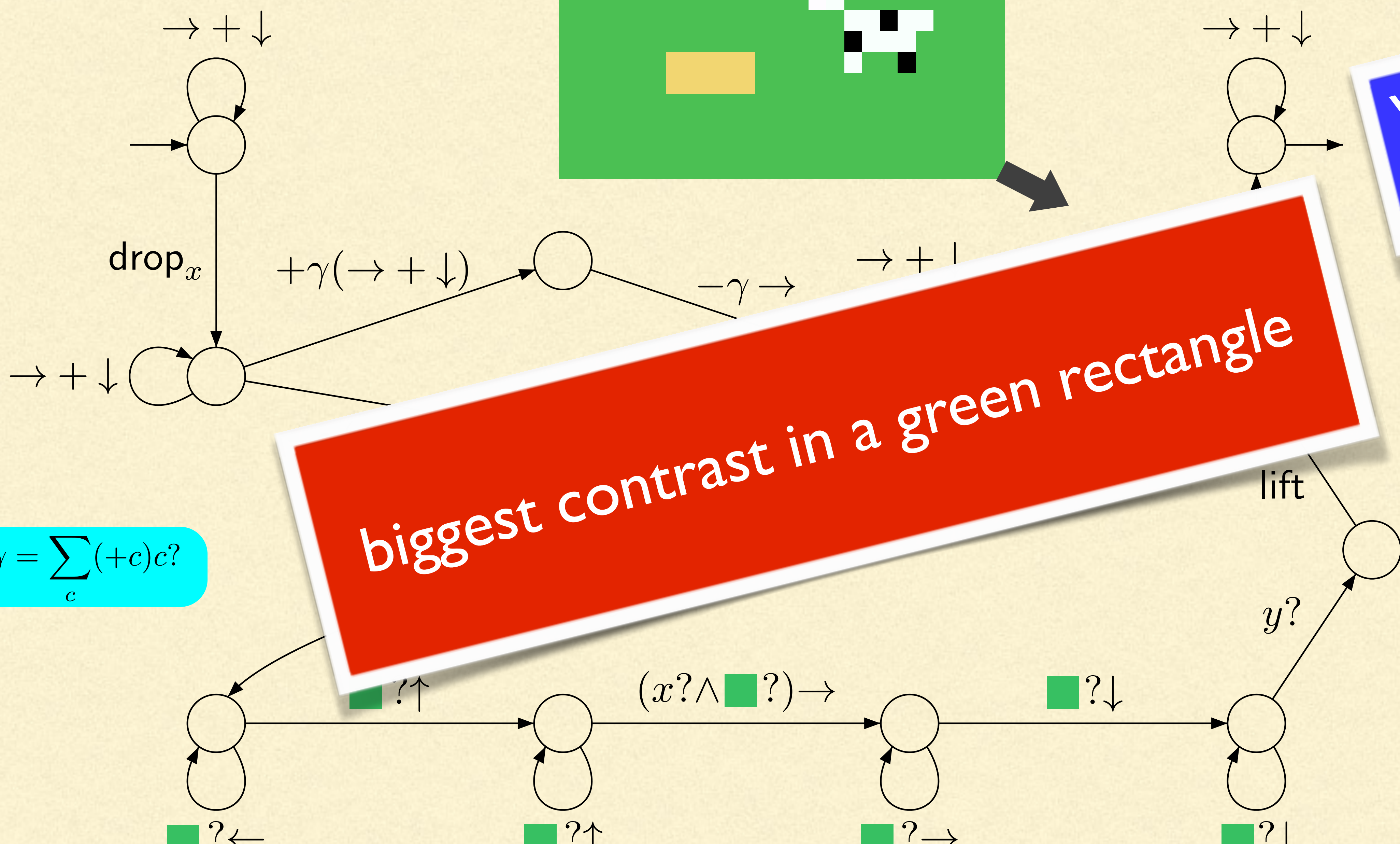
PEBBLE WALKING AUTOMATA



$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

0 , … , 0 , +255 , -0 , 0 , … , 0

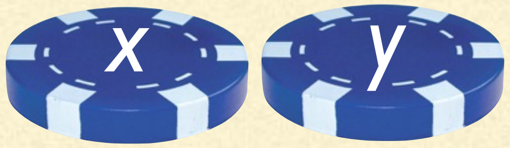
Value of the run: 255

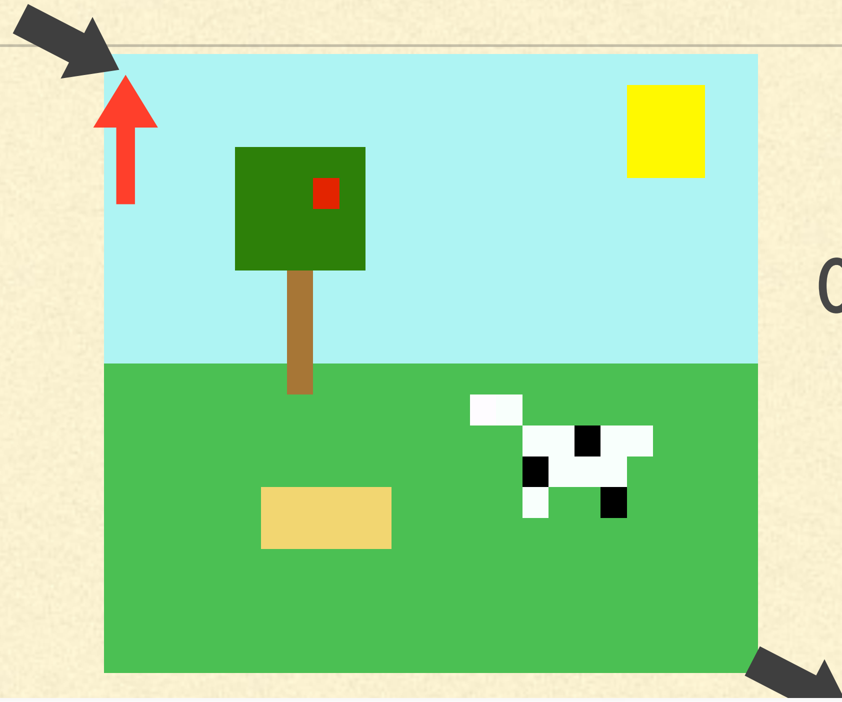


biggest contrast in a green rectangle

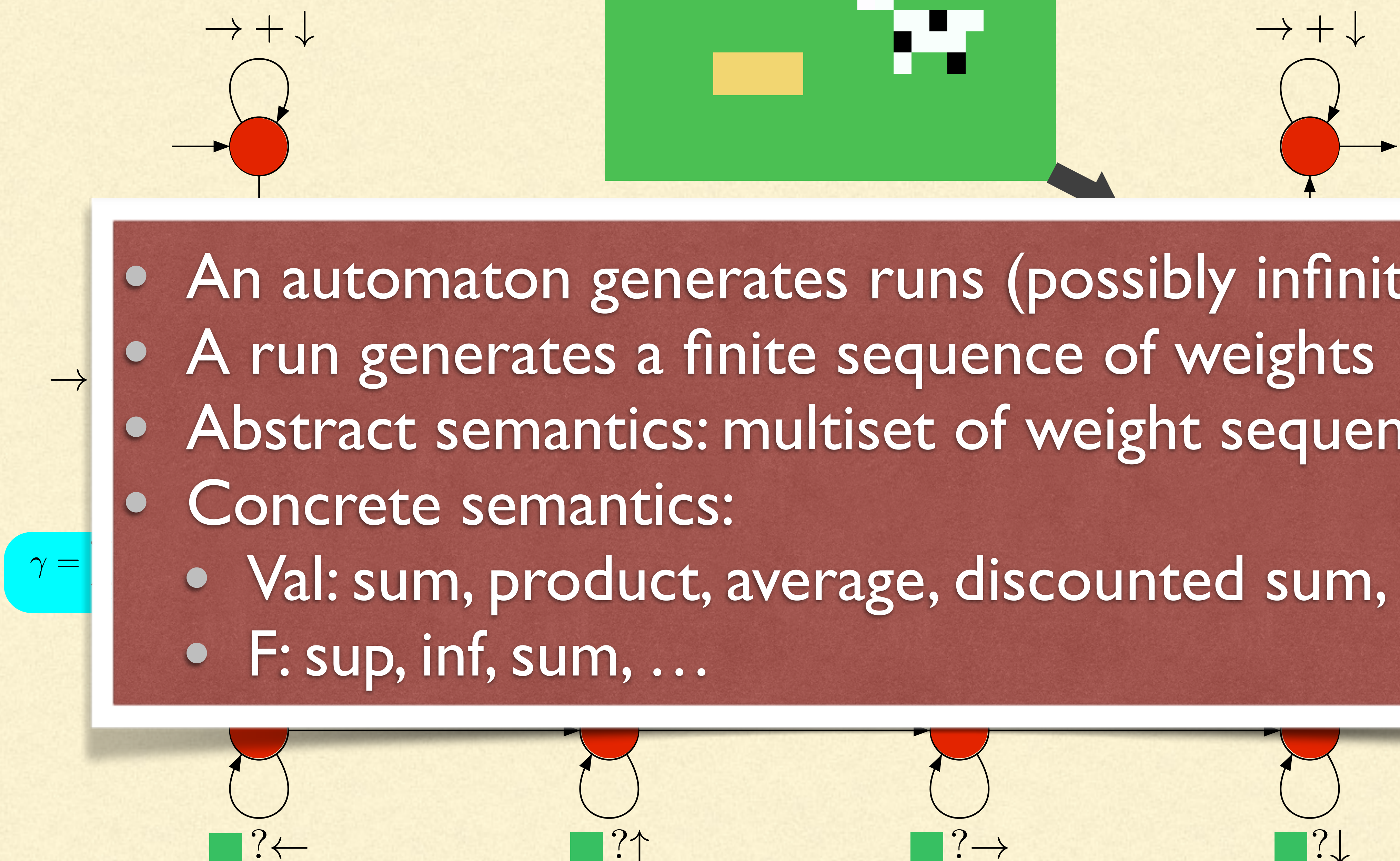
$\gamma = \sum_c (+c)c?$

PEBBLE WALKING AUTOMATA





0 , … , 0 , +255 , -0 , 0 , … , 0



- An automaton generates runs (possibly infinitely many)
- A run generates a finite sequence of weights
- Abstract semantics: multiset of weight sequences
- Concrete semantics:
  - Val: sum, product, average, discounted sum, …
  - F: sup, inf, sum, …

PEBBLE WALKING AUTOMATA

We cannot compute |w|! or $2^{|w|^2}$ with a WA

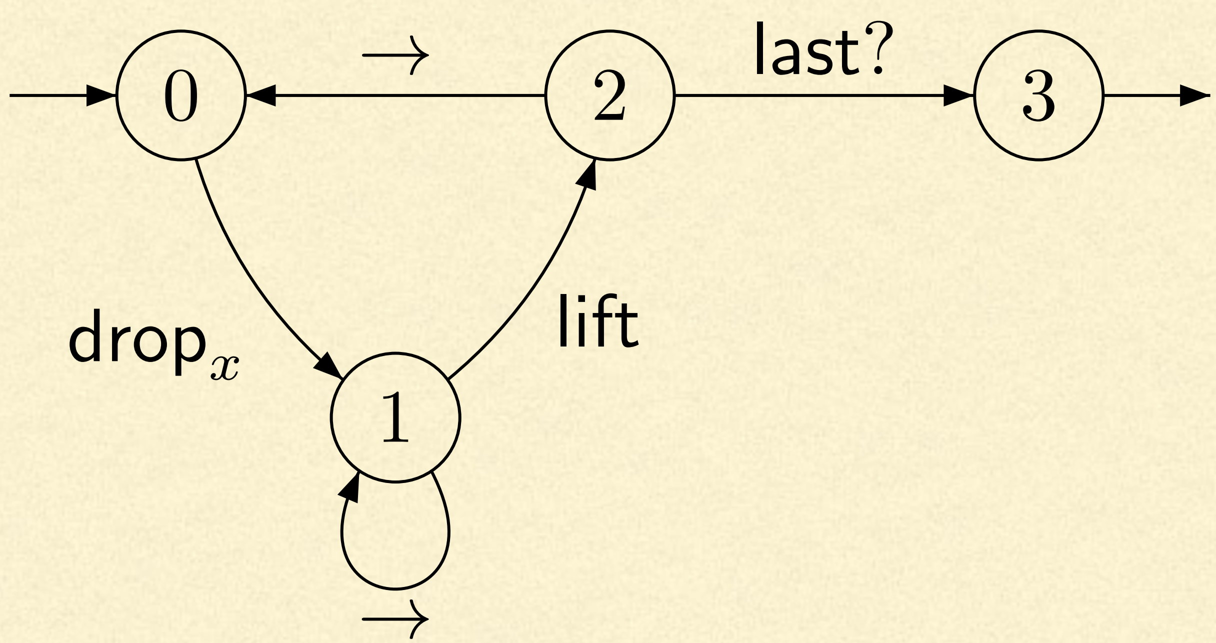
Let $\mathcal{A}$ be a WA over $(\mathbb{Q}, +, \times, 0, 1)$.

We have $|[\![\mathcal{A}]\!](w)| \leqslant k^{|w|}$ for all $w \in \Sigma^+$.

PEBBLE WALKING AUTOMATA

We cannot compute |w|! or $2^{|w|^2}$ with a WA

More paths: increased expressive power

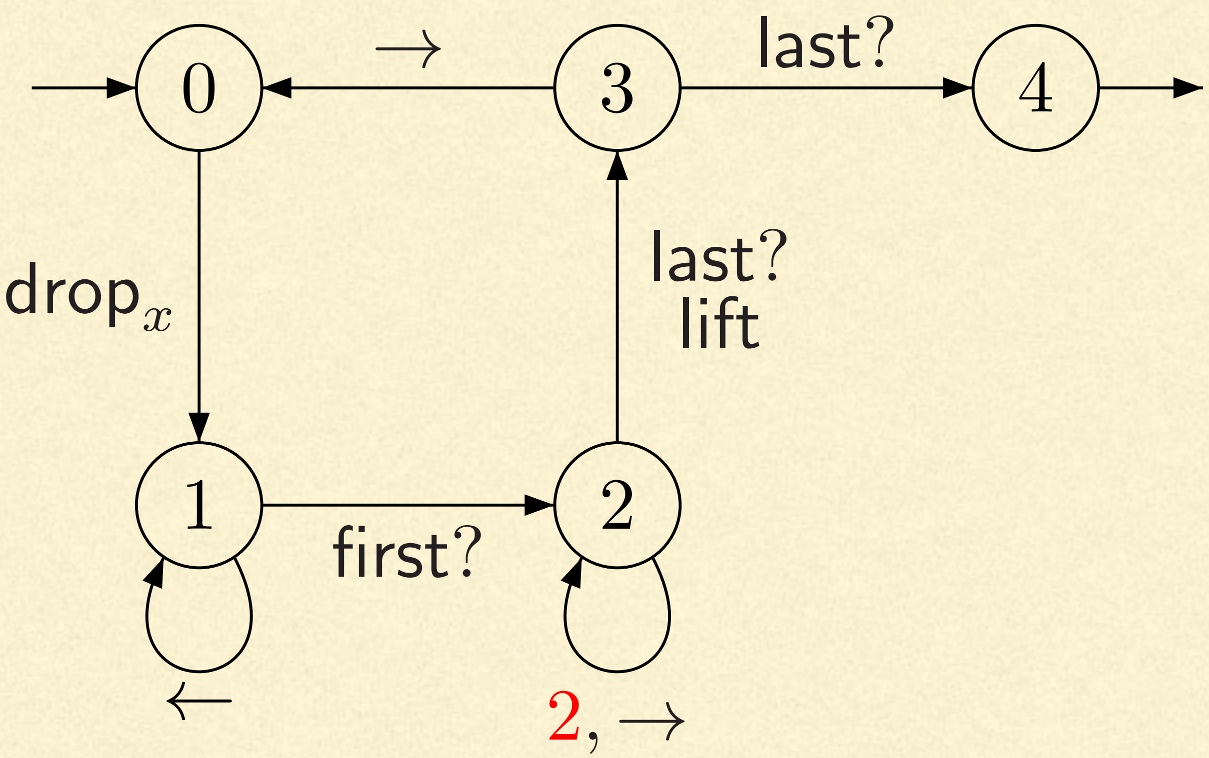


PEBBLE WALKING AUTOMATA

We cannot compute |w|! or $2^{|w|^2}$ with a WA

More paths: increased expressive power

Longer paths: increased expressive power



SEMIRINGS AND FIELDS

1. Matrix representation
2. Efficient evaluation
3. Rational expressions
4. Basis and applications: Decidability emptiness, equality
5. Basis and applications: Reductions
6. Basis and applications: Minimization

WEIGHT DOMAIN: SEMIRINGS

$$(\mathbb{S}, +, \times, 0, 1)$$

- $+$: associative and commutative, with neutral element 0
- $\times$: associative, with neutral element 1, distributive over addition
- 0: zero of the multiplicative operation

$$(\mathbb{R}, +, \times, 0, 1)$$
$$(\mathbb{Q}, +, \times, 0, 1)$$
$$(\mathbb{Z}, +, \times, 0, 1)$$
$$(\mathbb{N}, +, \times, 0, 1)$$

$$(\{0,1\}, \vee, \wedge, 0, 1)$$
$$([0,1], \max, \min, 0, 1)$$
$$(\mathbb{R} \cup \{-\infty, +\infty\}, \max, \min, -\infty, +\infty)$$

$$(\mathcal{P}(A^*), \cup, \cdot, \varnothing, \{\varepsilon\})$$
$$(\mathrm{Rat}(A^*), \cup, \cdot, \varnothing, \{\varepsilon\})$$

$$(\mathbb{R} \cup \{+\infty\}, \min, +, +\infty, 0)$$
$$(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$$

MATRICES OVER A SEMIRING

- Product of matrices:

Let $A \in S^{n \times m}$ and $B \in S^{m \times \ell}$

$$C = A \times B \qquad c_{i,j} = \sum_k a_{i,k} b_{k,j}$$

$$(S^{n \times n}, \times, \mathsf{Id}) \text{ is a monoid}$$

REPRESENTATION

$$F = (\lambda, \mu, \gamma)$$

row vector

morphism

column vector

$$\mu : \Sigma^* \to S^{n \times n}$$

$$F(w) = \lambda \times \mu(w) \times \gamma$$

Representation: Example

$\Sigma = \{0,1\}$ $w = 10010$ $\overline{w}^2 = 2^4 + 2^1 = 18$

Semiring: $(\mathbb{N}, +, \times, 0, 1)$ $\mu(0) = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ $\mu(1) = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$

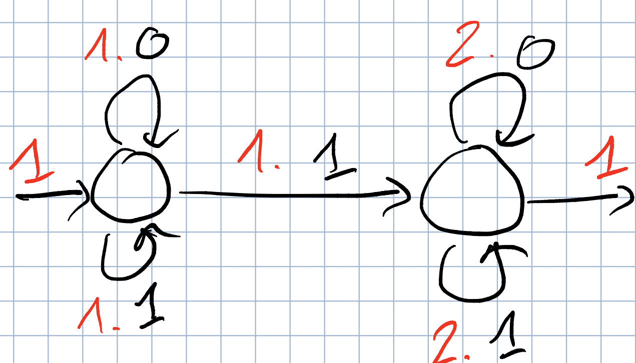
Claim: $\mu(w) = \begin{pmatrix}1 & \overline{w}^2\\ 0 & 2^{|w|}\end{pmatrix}$ proof: induction.

$\mu(\varepsilon) = Id = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ ✓

$$\mu(w1) = \begin{pmatrix}1 & \overline{w}^2\\ 0 & 2^{|w|}\end{pmatrix} \times \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 1+2\,\overline{w}^2\\ 0 & 2^{1+|w|}\end{pmatrix} = \begin{pmatrix}1 & \overline{w1}^2\\ 0 & 2^{|w1|}\end{pmatrix}$$ ✓

$$\mu(w0) = \begin{pmatrix}1 & \overline{w}^2\\ 0 & 2^{|w|}\end{pmatrix} \times \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 2\,\overline{w}^2\\ 0 & 2^{1+|w|}\end{pmatrix} = \begin{pmatrix}1 & \overline{w0}^2\\ 0 & 2^{|w1|}\end{pmatrix}$$ ✓

with $\lambda = (1, 0)$ $\gamma = \begin{pmatrix}0\\ 1\end{pmatrix}$ we get $\lambda\, \mu(w)\, \gamma = \overline{w}^2$ ✓



WA = REPRESENTATIONS

Let $\mathcal{A} = (Q, \Sigma, \Delta, I, F, \mathsf{wgt})$ be a WA over the semiring S

Define the morphism $\mu \colon \Sigma^* \to S^{Q \times Q}$ by $\mu(a)_{p,q} = \mathsf{wgt}(p, a, q)$

Claim: $\mu(w)_{p,q} = \sum_{p \xrightarrow{w} q} \mathsf{wgt}(p \xrightarrow{w} q)$

Define $\lambda_p = \begin{cases} 1 & \text{if } p \in I \\ 0 & \text{otherwise} \end{cases}$ and $\lambda_q = \begin{cases} 1 & \text{if } p \in F \\ 0 & \text{otherwise} \end{cases}$

$$\lambda \times \mu(w) \times \gamma = \sum_{\substack{p \xrightarrow{w} q \\ \text{accepting}}} \mathsf{wgt}(p \xrightarrow{w} q) = [\![\mathcal{A}]\!](w)$$

Representation and WA.

Claim: $\mu(w)_{p,q} = \sum_{p \xrightarrow{w} q} wgt(p \xrightarrow{w} q)$

Proof by induction.

$\mu(\varepsilon) = Id.$ $\mu(\varepsilon)_{p,q} = \begin{cases} 1 & \text{if } p=q \\ 0 & \text{otherwise} \end{cases}$ | if $p=q$: 1 run $wgt(p \xrightarrow{\varepsilon} p) = 1$; if $p \neq q$: No runs $\sum_{\emptyset} = 0$ ✓

$\mu(wa) = \mu(w) \times \mu(a)$

$$\begin{aligned}
\mu(wa)_{p,q} &= \sum_r \mu(w)_{p,r} \cdot \mu(a)_{r,q} \\
&= \sum_r \sum_{p \xrightarrow{w} r} wgt(p \xrightarrow{w} r) \cdot wgt(r \xrightarrow{a} q) \\
&= \sum_r \sum_{p \xrightarrow{w} r} wgt(p \xrightarrow{w} r \xrightarrow{a} q) \\
&= \sum_{p \xrightarrow{wa} q} wgt(p \xrightarrow{wa} q)
\end{aligned}$$

BOOLEAN SEMIRING

$$\mu(w)_{p,q} = \begin{cases} 1 & \text{if there exists a path } p \xrightarrow{w} q \\ 0 & \text{otherwise} \end{cases}$$

- Representation = transition monoid
- Rec. by automata = Rec. by morphisms

SEMIRINGS AND FIELDS

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Evaluation Problem (Boolean)

Given automaton (specification) $\mathcal{A}$ and word (model) w, compute $[\![\mathcal{A}]\!](w)$.

Try to optimize for small specifications and huge models.

DFA $\quad \mathcal{O}(|w|)$

One register: state reached after prefix u of w

$$\iota \xrightarrow{u} s \xrightarrow{a} \delta(s,a) \qquad \mathcal{O}(1)$$

NFA $\quad \mathcal{O}(|Q|^2|w|)$

$n = |Q|$ Boolean registers: $S = (s_1, \ldots, s_n) \in \mathbb{B}^{1\times n}$

After reading prefix u of w, register $s_q = 1$ if $\exists\ \iota \xrightarrow{u} q$ with ι initial.

$$\boxed{I} \xrightarrow{u} \boxed{S} \xrightarrow{a} S' = \boxed{S} \times \boxed{M(a)} \qquad \mathcal{O}(n^2)$$

where $M(a) \in \mathbb{B}^{n\times n}$ transition matrix for letter a.

Alternative: Determinize then evaluate. $\quad \mathcal{O}(2^{|Q|} + |w|)$

Evaluation Problem (Weighted 1-way)

1-WA $\mathcal{O}(|Q|^2|w|)$

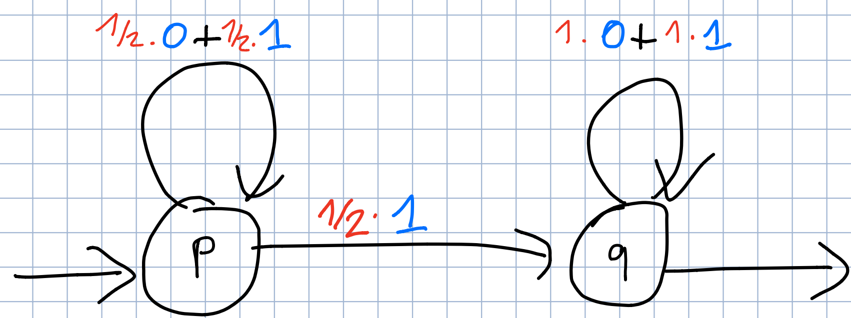
$n = |Q|$ quantitative registers: $S = (s_1, \ldots, s_n) \in \mathbb{S}^{1\times n}$

After prefix u of w, register $s_q = \sum_\rho \text{weight}(\rho)$ where $\rho \colon \iota \xrightarrow{u} q$ with ι initial.

$$I \xrightarrow{u} S \xrightarrow{a} S' = S \times M(a) \qquad \mathcal{O}(n^2)$$

where $M(a) \in \mathbb{S}^{n\times n}$ weighted transition matrix for letter a.

The matrix computation



$$\mu(0)=\begin{pmatrix}1/2 & 0\\ 0 & 1\end{pmatrix} \quad \mu(1)=\begin{pmatrix}1/2 & 1\\ 0 & 1\end{pmatrix} \quad \lambda=(1,0) \quad \gamma=\begin{pmatrix}0\\ 1\end{pmatrix}$$

$w = 01011$

λ	$\mu(0)$	$\mu(1)$	$\mu(0)$	$\mu(1)$	$\mu(1)$
	$\begin{pmatrix}1/2 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1/2 & 1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1/2 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1/2 & 1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1/2 & 1\\ 0 & 1\end{pmatrix}$
$(1,0)$	$(1/2,0)$	$(1/4, 1/2)$	$(1/8\ 1/2)$	$(1/16, 1/2+1/8)$	$(1/32, 1/2+1/8+1/16)$

Claim: $\lambda\mu(w)=\left(\frac{1}{2^{|w|}}, \overline{0.w}^2\right)$

Proof by induction.

<u>$w=\varepsilon$</u>: $\lambda\mu(w)=\lambda\times \mathrm{Id}=\lambda=(1,0)$ ✓

<u>$w.0$</u>:

$$\lambda\mu(w0)=\lambda\mu(w)\mu(0)=\left(\frac{1}{2^{|w|}}, \overline{0.w}^2\right)\cdot\begin{pmatrix}1/2 & 0\\ 0 & 1\end{pmatrix}=\left(\frac{1}{2^{|w|+1}}, \overline{0.w}^2\right) ✓$$

<u>$w.1$</u>: $$\left(\frac{1}{2^{|w|}}, \overline{0.w}^2\right)\times\begin{pmatrix}1/2 & 1\\ 0 & 1\end{pmatrix}=\left(\frac{1}{2^{|w|+1}}, \overline{0.w}^2+\frac{1}{2^{|w|+1}}\right) ✓$$

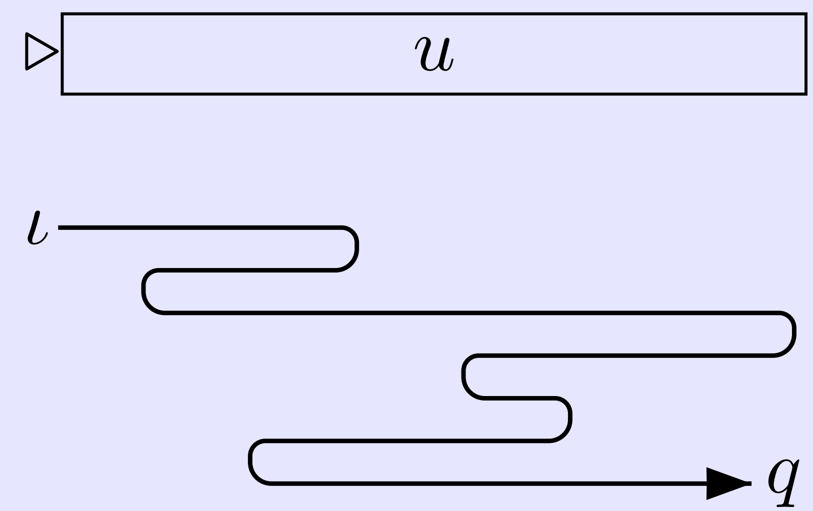
Evaluation Problem (Weighted 2-way)

2-WA $\mathcal{O}(|Q|^3|w|)$

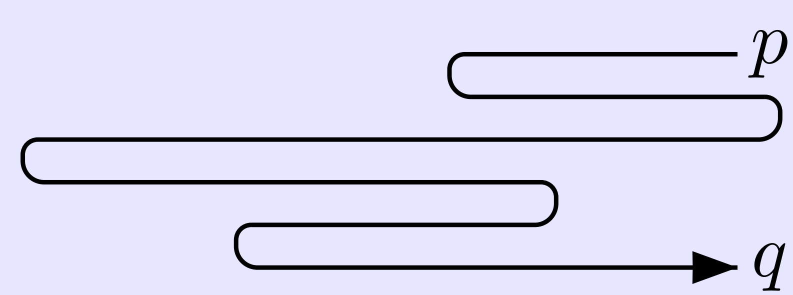
$n + n^2$ quantitative registers: $S = (s_1, \ldots, s_n) \in \mathbb{S}^{1\times n}$ and $C = (c_{p,q}) \in \mathbb{S}^{n\times n}$

After prefix u of w,

$s_q = \sum_\rho \text{weight}(\rho)$ where ρ:



$c_{p,q} = \sum_\rho \text{weight}(\rho)$ where ρ:



WEIGHT DOMAINS: CONTINUOUS SEMIRINGS

$$(\mathbb{S}, +, \times, 0, 1)$$

every infinite sum exists and is the limit of finite *approximate* sums, keeping good properties of usual semiring

$(\mathbb{R}, +, \times, 0, 1)$
$(\mathbb{Q}, +, \times, 0, 1)$
$(\mathbb{Z}, +, \times, 0, 1)$
$(\mathbb{N}, +, \times, 0, 1)$

$(\{0,1\}, \vee, \wedge, 0, 1)$
$([0,1], \max, \min, 0, 1)$
$(\mathbb{R} \cup \{-\infty, +\infty\}, \max, \min, -\infty, +\infty)$

$(\mathbb{R}^+ \cup \{+\infty\}, +, \times, 0, 1)$
$(\mathbb{N} \cup \{+\infty\}, +, \times, 0, 1)$

$\mathbb{R} \cup \{-\infty, +\infty\}$
$(\mathbb{R} \cup \{+\infty\}, \min, +, +\infty, 0)$
$(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$

$(\mathcal{P}(A^*), \cup, \cdot, \varnothing, \{\varepsilon\})$
~~$(\mathrm{Rat}(A^*), \cup, \cdot, \varnothing, \{\varepsilon\})$~~

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Assume S is a field (or a subsemiring of a field)

Let $F : \Sigma^* \to S$ defined by a representation (λ, μ, γ) of dimension n : $\lambda \in S^{1 \times n}$ $\mu(a) \in S^{n \times n}$ $\gamma \in S^{n \times 1}$

Consider the vector space $E = \langle \lambda . \mu(\Sigma^*) \rangle \subseteq S^n$

$E = \langle \{ \lambda \mu(w) \mid w \in \Sigma^* \} \rangle$

Thm: We can compute a basis B of E in time $O(|\Sigma| . n^3)$

Cor: Decidability of the emptiness problem : $F = 0$?

Decidability of the equivalence problem : $F = G$?

Proof: 1) Compute B

Check $x . \gamma = 0 \quad \forall x \in B$

2) Let $F = (\lambda_1, \mu_1, \gamma_1)$ $G = (\lambda_2, \mu_2, \gamma_2)$

Define $F - G$ by (λ, μ, γ) where

$\lambda = (\lambda_1, -\lambda_2)$ $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ and $\mu(a) = \left(\begin{array}{c|c} \mu_1(a) & 0 \\ \hline 0 & \mu_2(a) \end{array} \right)$

Check whether $F - G = 0$. $\square$

Algorithm computing the basis

$n \times |\Sigma| \times n^2$

Assume $x_1 = \lambda = \lambda\mu(\varepsilon) \neq 0$

$B \leftarrow \{x_1\}$; $Todo \leftarrow \{x_1\}$

While $Todo \neq \emptyset$ Do   Inv: B is free $\wedge$ $\langle B \rangle \subseteq \langle \lambda\mu(\Sigma^*) \rangle$

  Take & Remove x in Todo

  For each $a \in \Sigma$ Do

    If $y = x.\mu(a) \notin \langle B \rangle$ then Add y to B and Todo fi

  End For

End While

Termination: $\dim \langle \lambda\mu(\Sigma^*) \rangle \leq n$

Hence we add at most n vectors in B and Todo

Fact: $\forall x \in B \ \forall a \in \Sigma \quad x.\mu(a) \in \langle B \rangle$

Claim: At the end $\langle B \rangle = \langle \lambda\mu(\Sigma^*) \rangle$

Proof: Induction on $|w|$

- $w = \varepsilon$ ✓ $\lambda = \lambda\mu(\varepsilon) \in B$
- wa $\lambda\mu(wa) = \lambda\mu(w).\mu(a) = \sum_{i=1}^{k} \alpha_i x_i . \mu(a)$ $B = \{x_1, \ldots, x_k\}$

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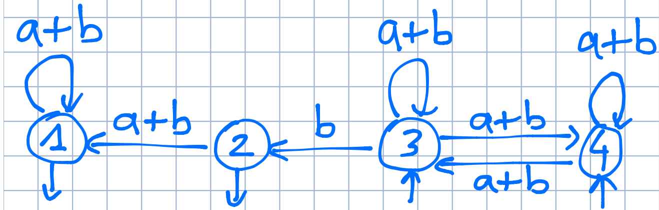
Reduction of a representation



$\Sigma = \{a, b\}$ $\mathbb{N}$ all coeff 1

$$\mu(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 \quad (0\ 0\ 1\ 1)$

$x_1 \cdot \mu(a) \quad (0\ 0\ 2\ 2) = 2 \cdot x_1$

$x_1 \mu(b) \quad (0\ 1\ 2\ 2) = 2x_1 + x_2$

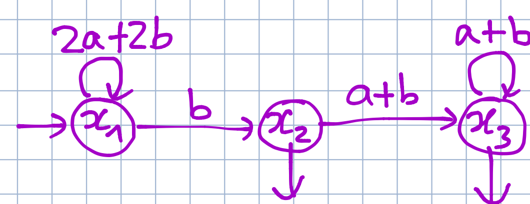
$x_2 \mu(a) \quad (1\ 0\ 0\ 0) = x_3$

$x_2 \mu(b) = x_3$

$x_3 \mu(a) = x_3 = x_3 \mu(b)$

Basis

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = X$$



$$\mu'(a) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} \qquad \mu'(b) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$X \mu(a) = \mu'(a) X$

$X \mu(b) = \mu'(b) X$

$$\lambda' = (1\ 0\ 0) \qquad \gamma' = X\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Thm: $(\lambda, \mu, \gamma) \cong (\lambda', \mu', \gamma')$

<u>Claim</u> $\forall w \in \Sigma^* \quad \lambda' . \mu'(w) . X = \lambda . \mu(w)$

By induction on $|w|$

$w = \varepsilon \quad \lambda' \mu'(\varepsilon) = \lambda' . Id = \lambda' = (1\ 0\ 0) \qquad \lambda' . X = x_1 = \lambda = \lambda \mu(\varepsilon)$

<u>wa</u>: $\lambda \mu(wa) = \lambda \mu(w) \mu(a) = (\lambda' \mu'(w))(X \mu(a)) = \lambda' \mu'(w) \mu'(a) X$ ✓

Rem: $X \mu(a) = \mu'(a) X$ by definition of μ' ✓

<u>Cor</u>: $\lambda \mu(w) \gamma = (\lambda' \mu'(w))(X . \gamma) = \lambda' \mu'(w) \gamma'$ ✓

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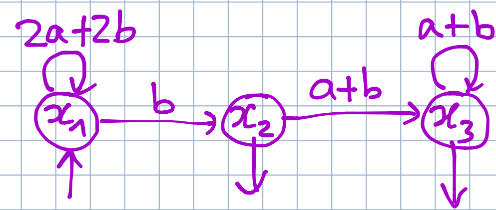
Transpose and mirror

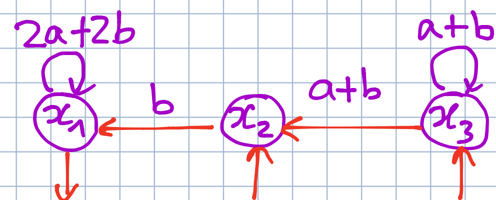
Let $F : \Sigma^* \to S$ presented by $\mathcal{A} = (\lambda, \mu, \gamma)$

Transpose $\mathcal{A}^t = (\gamma^t, \mu^t, \lambda^t)$

Example $\mathcal{A}$:



$\mathcal{A}^t$:



Prop: $[\![\mathcal{A}]\!](w) = [\![\mathcal{A}^t]\!](\tilde{w})$

Proof 1) bijection between

Runs $p \xrightarrow{w} q$ in $\mathcal{A}$ and Runs $q \xrightarrow{\tilde{w}} p$ in $\mathcal{A}^t$ ✓

2) $w = a_1 a_2 \cdots a_k$

$$[\![\mathcal{A}]\!](w) = [\![\mathcal{A}]\!](w)^t = \left(\lambda . \mu(a_1) \cdots \mu(a_k) . \gamma\right)^t$$

$$= \gamma^t . \mu(a_k)^t \cdots \mu(a_1)^t . \lambda^t = [\![\mathcal{A}^t]\!](\tilde{w})$$ ✓

Minimisation:

Input $\mathcal{A} = (\lambda, \mu, \gamma)$

Transpose $\mathcal{A}$: $\mathcal{A}^t = (\gamma^t, \mu^t, \lambda^t)$

Reduce $\mathcal{A}^t \longrightarrow \mathcal{B} = (\lambda_1, \mu_1, \gamma_1)$

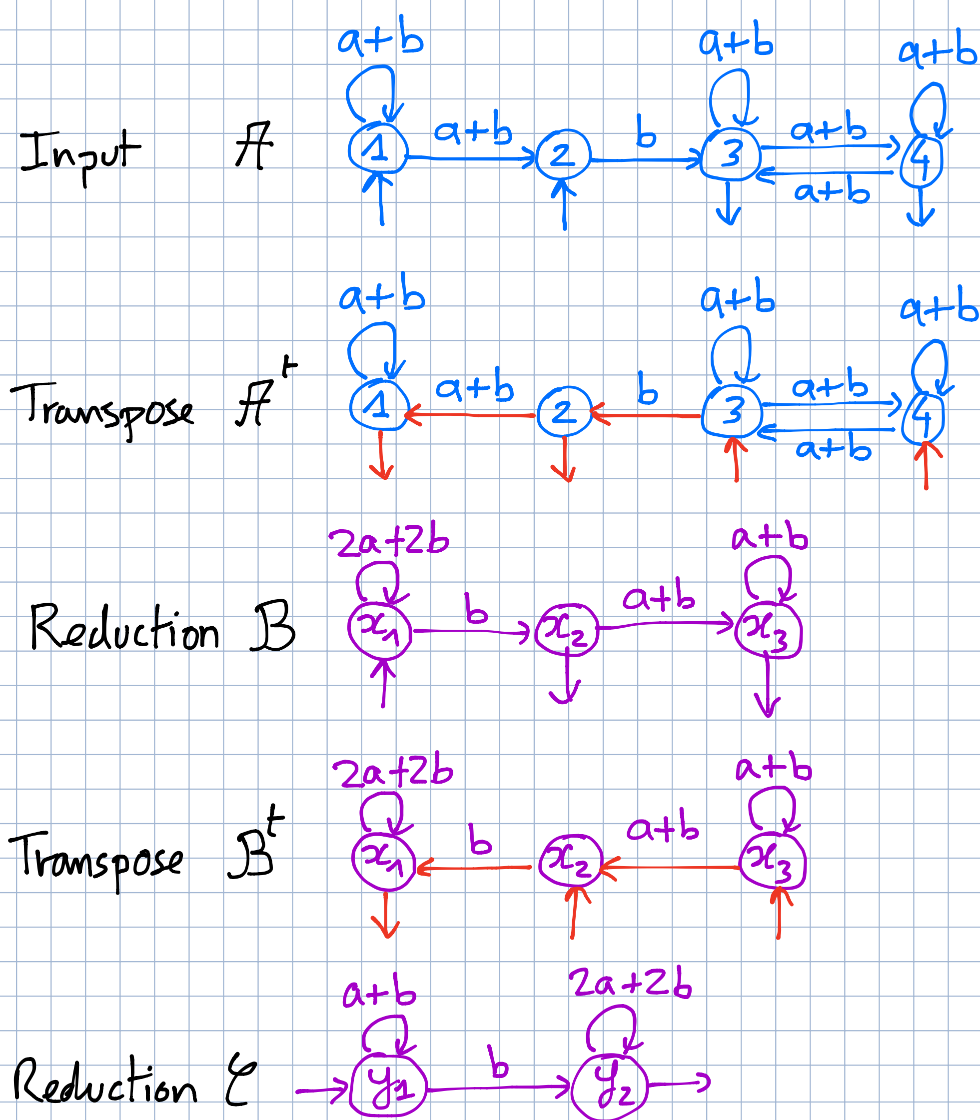
Transpose $\mathcal{B}$: $\mathcal{B}^t = (\gamma_1^t, \mu_1^t, \lambda_1^t)$

Reduce $\mathcal{B}^t \longrightarrow \mathcal{C} = (\lambda_2, \mu_2, \gamma_2)$

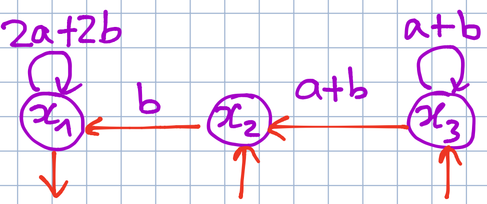
Fact: $[\![\mathcal{A}]\!] = [\![\mathcal{C}]\!]$

Thm: $\mathcal{C}$ is minimal.

Minimisation Example



$\mathcal{B}^t$



$$\mu(a) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mu(b) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$y_1 \cdot \mu(a) = (0\ 1\ 1) = y_1$

$y_1 \mu(b) = (1\ 1\ 1) = y_1 + y_2$

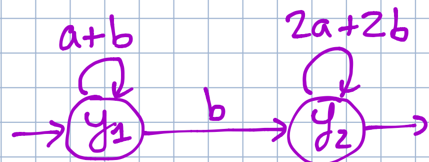
$y_2 \mu(a) = (2, 0, 0) = 2\, y_2$

$y_2 \mu(b) = (2, 0, 0) = 2\, y_2$

<u>Basis</u> Y $\begin{matrix} y_1 \\ y_2 \end{matrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$\mu'(a) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu'(b) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\lambda' = (1, 0) \qquad \gamma' = Y\gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



THANK YOU