An Efficient Computation of Structural Relations in Unary Regular Nets.

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Abstract

In this paper, we present a new technique to compute structural relations at the colored net level. It consists in defining the relations in a symbolic way, i.e., without refering to the colors of the nodes of the net. This definition is useful only provided that the operations on the symbolic relations can be performed without an increased cost with respect to the operations on ordinary relations. We show that this condition is fulfilled if the color functions of the net respect some stability conditions. We apply our method to the structural conflict and show that it can be extended to such algorithms on a net as the transitive closure of a relation.

1. INTRODUCTION

Using Petri nets for modelling computer systems allows us to check directly on the model such properties as mutual exclusion or conflict of actions. Checking those properties on the set of reachable states would be possible, but also very expensive. Working directly on the model is much cheaper.

However, modelling complex systems with Petri nets quickly results in inextricable models. High-level net models have thus been proposed, where information is added to the tokens of the Petri net. This information is taken into account by replacing the integer valuations on the arcs of the nets by functions that indicate which tokens must be selected for the firing. The description of the system is more concise, although the same features are modelled. Since high-level nets - Predicate/Transition nets [Gen 91] or Colored nets [Jen 91] - have been introduced, a lot of research has been done to extend to them the results on ordinary Petri nets. But the main difficulty is to use the conciseness of the model in the analysis process : having a compact model does not mean that we can perform an efficient analysis.

In fact, the problem is to analyze the model without refering to the equivalent unfolded net, i.e., an ordinary Petri net with the same behavior as the high-level net. Such analysis methods have already been applied to net transformations [Had 91]. In this paper, we present an extension of those results to the computation of structural relations in unary regular nets, which are a subclass of colored nets.

This extension is threefold. We first show how to represent at the colored net level the relations between the nodes of the unfolded net : these relations are given by functions relating the powersets of the color domains of two nodes. As the computation of structural properties relies on the study of the graph underlying the Petri net model, we then adapt elementary operations to the handling of our symbolic

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relations. Finally, we show that this process can be generalized to the derivation of efficient graph algorithms on the colored net. As an example, we extend the transitive closure algorithm of Warshall.

The reason why we chose unary regular nets (URNs) as a model is that if we want to obtain efficient algorithms, i.e., better than algorithms computing the same property on the unfolded net, the color functions of the model must satisfy some stability conditions. We believe that these conditions are satisfied by models as general as Well-Formed nets [Chi 91]. But for simplicity reasons, we will limit our study to the simpler model of URNs.

The paper is organized as follows. Section 2 presents unary regular nets. Section 3 shows how simple relations can be computed at the net level by using operations on powersets. Section 4 explains the problem of extending these relations to indirect relations and shows on the transitive closure algorithm that it is still possible to obtain efficient results. The last section presents some perspectives to this work.

2. UNARY REGULAR NETS

Unary Regular nets [Had87] are a subclass of colored nets, whose nodes all have the same color domain, and whose color functions are defined as linear combinations of two basic color functions, the identity function X and the diffusion function S that synchronizes all the elements of a set E.

Notation : Throughout the paper, we will use functions and operations on multisets. Intuitively, a multiset is a set in which elements can appear several times. A multiset y on a set E can be represented in a formal way by a sum :

$$y = \Sigma_{c \in E}\, y(c).c$$

where y(c) is the number of occurrences of c in the multiset y. We will use E_{MS} to denote the set of multisets on E.

Definition 2.1 Let X and S be two functions mapping a set E on E_{MS}. These functions are defined by :

$$X(c) = c, \quad S(c) = \Sigma_{c' \in E}\, c'$$, which in fact defines a constant function.

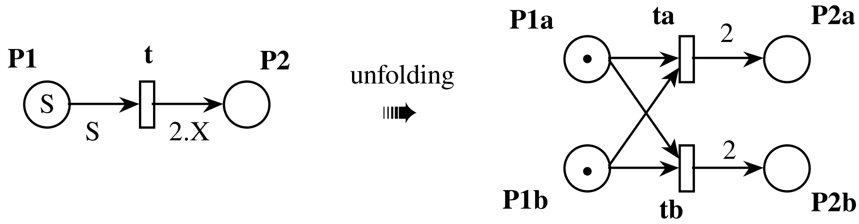
The complete formal definition of the model is the following.

Definition 2.2 A unary regular net $N = \langle P, T, E, W^-, W^+, H, M_0 \rangle$ is defined by :

- P the set of places
- T the set of transitions with $P \cup T \neq \emptyset$ and $P \cap T = \emptyset$
- E is some finite non-empty set. An item of E is called a color and for all s in $P \approx T$, E is the color domain of s.
- W^- and W^+ are the input and output functions defined on P x T. $W^-(p, t)$ (resp. $W^+(p, t)$) is a function from E_{MS} to E_{MS}, and is defined as a linear sum a.S + b.X with $a \geq 0$ and $a+b \geq 0$.
- M_0 the initial marking of the net is a vector of P.

We can compare on an example the URN model with the corresponding unfolded net, i.e., an ordinary Petri net obtained by creating one place (resp. one transition) for each possible couple (p, c), $c \in E$ (resp. (t, c), $c \in E$). An arc between (p, c) and (t, c') (resp. between (t, c') and (p, c)) exists with a valuation k if the input (resp. output) function is such that $W^-(p, t)(c')(c) = k$ (resp. $W^+(p, t)(c')(c) = k$). The initial marking of (p, c) is the number of color c tokens that p contains in the colored net.

Example : Let us consider an URN with E = {a, b}. The following model can be unfolded in an ordinary Petri net with 4 places and 2 transitions.



Definition 2.3 A transition t is enabled for a marking M and a colour $c_t \in E$ if and only if :

$$\forall p \in P, \quad M(p) \geq W^-(p,t)(c_t)$$

The firing of t for a marking M and a colour $c_t \in E$ gives the marking M' defined by :

$$\forall p \in P, \quad M'(p) = M(p) - W^-(p,t)(c_t) + W^+(p,t)(c_t)$$

We want to obtain directly at the colored net level simple or complex relations between the elements of the net. We call simple a relation that concerns two transitions sharing some common input tokens, e.g., structural conflict. In a colored net, transitions that share an input place may require tokens of different colors for their firings, in which case they will not be conflicting. Hence, the analysis of structural relations is directly connected to the study of the color functions labelling the arcs of the net.
Simple relations can be extended to complex relations, such as indirect conflict, involving transitions that do not directly share input tokens, but that are related to each other through some other transitions. For instance if a transition t_1 shares an input place with t_2 and t_2 shares an input place with t_3, then t_1 may be in indirect conflict with t_3. Hence, we must be able to compute the transitive closure of a simple relation.

In order to study structural relations at the colored net level, we must find for them an expression that does not refer to the colors of the nodes of the net. We introduce this expression in the next section through the example of structural conflict. And we show that, provided that the color functions of the net verify some stability conditions, these relations can be computed very efficiently.

3. SIMPLE RELATIONS : THE CASE OF STRUCTURAL CONFLICT

3.1. Introductive Example

We consider that two transitions are in structural conflict if they share some input token. This is a simplified definition of the relation, but it provides a simple example. A more detailed analysis can be found in [Dut 92]. In Figure 2(a), t_1 is in structural conflict with t_2, t_2 with t_1 and t_3, and t_3 with t_2.

To extend this relation to URNs, we must not only consider the transitions but also their color instances. Let us focus on transition t_2 in the net in Figure 2(b). (t_2, c) is enabled if and only if a token c is present in both P1 and P2. Thus, to compute the structural conflict relation, we must know which instances of t_1 require a token c in P1, and which instances of t_3 require a token c in P2. The unfolding of the net will give the answer, i.e., (t_2, c) is in structural conflict with (t_1, c) and with all the color instances of t_3.

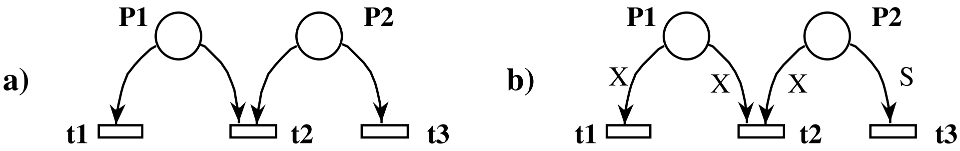


Figure 2 : Structural conflict in (a) ordinary Petri net and (b) URNs.

But by refering to the unfolded net, we lose all the advantage that could be taken from the structure of the colored net. In order to express such relations as structural conflict without refering to the colors of the nodes, we introduce the notion of symbolic relation, which can be seen as a folded expression of a relation. We justify our definition by showing that, if the color functions of the net verify some stability properties, the computation of a symbolic relation is not a complex operation.

3.2 Definition of Symbolic Relations

Symbolic relations on a colored net can be defined in a matricial way. As they account for a link between colors belonging to the color domains of two nodes of the net, we introduce $\mathcal{P}(E)$, the powerset of E, which is the set of subsets of E.

Definition 3.1 Let N be a URN. Let M be a square matrix indexed by the nodes of N such that M(s', s) is a function from $\mathcal{P}(E)$ to $\mathcal{P}(E)$. M is called a symbolic relation of N, and $\mathcal{R}_M$ denotes a relation between the nodes of the unfolded net, defined by :

$$(s', c')\ \mathcal{R}_M\ (s, c) \Leftrightarrow c' \in M(s', s)(c)$$

This definition can be applied to the expression of very simple relations, such as the successor relation. In the unfolded net, (s', c') is a successor of (s, c) if there is an arc from (s, c) to (s', c').

For instance, if s is a transition, the successors of (s, c) are given by $\{(s', c') \mid W^+(s', s)(c)(c') > 0\}$.
Hence, we introduce functions on powersets that allow us to define such relations.

Definition 3.2 Let f be a function mapping E_{MS} onto E_{MS}. $\overline{f} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is a function defined by :

$$\overline{f}(x) = \{\ y \ /\ f(x)(y) > 0\ \}$$

Applying this definition to the successor relation, if s is a transition we obtain $c' \in \overline{W^+(s', s)}\ (c)$

Consider now the case where s is a place. As a color function is defined from the domain of a transition to the domain of a place, the successors of (s, c) are given by $\{(s', c') \mid W^-(s, s')(c')(c) > 0\}$.
In this case, the former definition cannot be used directly to obtain the symbolic relation. We thus need to introduce the transpose of a function.

Definition 3.3 : Let f be a function mapping E_{MS} onto E_{MS}. Then f^t is a function from E_{MS} onto E_{MS} defined by : $f^t(c)(c') = f(c')(c)$
Let f be a function mapping $\mathcal{P}(E)$ onto $\mathcal{P}(E)$. Then f^t is a function mapping $\mathcal{P}(E)$ onto $\mathcal{P}(E)$ defined by :

$$y \in f(x) \Leftrightarrow x \in f^t(y)$$

We can prove very easily that for a function f mapping E_{MS} onto E_{MS}, $\overline{f^t} = \overline{f}^{\,t}$.

Hence, if s is a place, we obtain for the successor relation $c' \in \overline{W^-(s, s')}^{\,t}(c)$

Summarizing these results, if we call SUC the symbolic successor relation we have :

$\forall$ p, p' $\in$ P, $\forall$ t, t' $\in$ T SUC(p, p') = Ø   SUC(t, t') = Ø

SUC(t, p) = $\overline{W^-(p, t)}^{\,t}$ SUC(p, t) = $\overline{W^+(p, t)}$

However, in the case of general color functions, this definition of symbolic relation may not offer any improvement as the cost of computing $\overline{f}$ would be equivalent to the cost of unfolding the net. But for URNs, and more generally for well-structured color functions, $\overline{f}$ can be obtained very easily and moreover with a parametric definition (independent of the cardinalities of the color domains). The next proposition gives the values of $\overline{f}$ for URNs.

Proposition 3.1 : Let f = a.S + b.X be a function from E_{MS} onto E_{MS}. Then $f^t = f$ and $\overline{f}$ is given by :

a = b = 0 $\Rightarrow$ $\overline{f} = \overline{0}$ where $\overline{0}$: P(E) $\rightarrow$ P(E) is defined by $\overline{0}$ (c) = Ø

a = 0 and b > 0 $\Rightarrow$ $\overline{f} = \overline{X}$ where $\overline{X}$: P(E) $\rightarrow$ P(E) is defined by $\overline{X}$ (c) = {c}

a > 0 and a + b > 0 $\Rightarrow$ $\overline{f} = \overline{S}$ where $\overline{S}$: P(E) $\rightarrow$ P(E) is defined by $\overline{S}$ (c) = E

a > 0 and a + b = 0 $\Rightarrow$ $\overline{f} = \overline{S\text{-}X}$ where $\overline{S\text{-}X}$: P(E) $\rightarrow$ P(E) is defined by $\overline{S\text{-}X}$ (c) = E\{c}

3.3 Application to the case of structural conflict

In the same way as we have obtained the successors of a transition t_1, we could obtain its predecessors by substituting $\overline{W^-(p, t_1)}$ to $\overline{W^+(p, t_1)}$. Applying this function to some color c gives the set of colors of p that are inputs of (t_1, c). Computing the successors of the colors belonging to this set gives for any transition t_2 the instances which are in structural conflict with (t_1, c). Hence, the expression of the structural conflict relation is given by :

$$SC(t_2, t_1) = \bigcup_{p \in P} \left(\overline{W^-(p, t_2)}^{\,t} \circ \overline{W^-(p, t_1)}\right)$$

where the operations $\cup$ and o are classically defined by : $\forall$ f, g two functions P(E) $\rightarrow$ P(E),

$(f \cup g)(x) = f(x) \cup g(x)$ $(g \circ f)(x) = g[f(x)]$

Our method improves the results obtained on the unfolded net only if we can compute efficiently the composition and the union of functions. Hence, we require that the set of functions we consider be stable under the operations that we perform on the functions, i.e., that the result of the application of an operation to a subset of functions is a function of the set we consider.

Proposition 3.2 Let F = { $\overline{0}$, $\overline{X}$, $\overline{S}$, $\overline{S\text{-}X}$ }, then F is { $\cup$ o} - stable. The following tables give the results of the operations and show that the complexity of their computation is equivalent to the access of a bidimensional array.

$\cup$	$\overline{0}$	$\overline{X}$	$\overline{S}$	$\overline{S-X}$
$\overline{0}$	$\overline{0}$	$\overline{X}$	$\overline{S}$	$\overline{S-X}$
$\overline{X}$	$\overline{X}$	$\overline{X}$	$\overline{S}$	$\overline{S}$
$\overline{S}$	$\overline{S}$	$\overline{S}$	$\overline{S}$	$\overline{S}$
$\overline{S-X}$	$\overline{S-X}$	$\overline{S}$	$\overline{S}$	$\overline{S-X}$

$\circ$	$\overline{0}$	$\overline{X}$	$\overline{S}$	$\overline{S-X}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{X}$	$\overline{0}$	$\overline{X}$	$\overline{S}$	$\overline{S-X}$
$\overline{S}$	$\overline{0}$	$\overline{S}$	$\overline{S}$	$\overline{S}$
$\overline{S-X}$	$\overline{0}$	$\overline{S-X}$	$\overline{S}$	$\overline{S}^{\dagger}$

† this is true only if | C | > 2. If | C | = 2 the result is $\overline{X}$. If | C | = 1, the result is $\overline{0}$

If we apply this definition to our introductive example, we obtain the following results :

$SC(t_1, t_2) = SC(t_2, t_1) = \overline{X} \circ \overline{X} = \overline{X}$

$SC(t_2, t_3) = \overline{X} \circ \overline{S} = \overline{S}$ $\qquad$ $SC(t_3, t_2) = \overline{S} \circ \overline{X} = \overline{S}$

If we take for instance $SC(t_2, t_1)$, as $SC(t_2, t_1)(c)$ is the set of colors of t_2 that are in structural conflict with (t_1, c), we know that these colors are the elements of $\overline{X}(c)$, i.e., {c}. Hence, (t_1, c) is in structural conflict with (t_2, c).

4. INDIRECT RELATIONS : THE TRANSITIVE CLOSURE ALGORITHM

4.1. Introductive example : the case of indirect structural conflict

The indirect structural conflict is defined as the transitive closure of the structural conflict relation. In Figure 3(a), t_1 is in structural conflict with t_2 as they share P1 in input, t_2 is in structural conflict with t_3 as they share P3 in input, hence t_1 is in indirect structural conflict with t_3. In this case, P2 has no influence on the relation.

Consider now the URN in Figure 3(b). We assume E = {a, b, c}. In this net, (t_3, c) shares a token c in P3 with (t_2, a) and (t_2, b). (t_2, a) and (t_2, b) share the tokens a, b and c in P2 with (t_2, c), and (t_2, c) shares a token c in P1 with (t_1, c). Hence, (t_3, c) is in indirect structural conflict with (t_1, c), and in this case P2 acts on the relation as it introduces a correlation among the different colors of t_2.

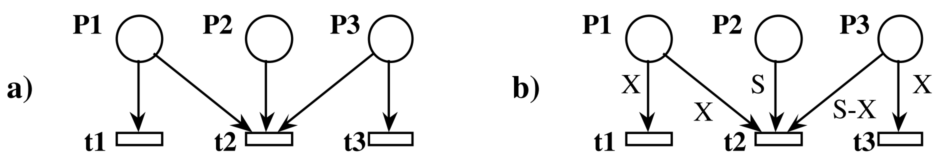


Figure 3 : Indirect structural conflict in (a) ordinary Petri nets and (b) URNs.

4.2. The transitive closure of a relation

We want an algorithm that operates on symbolic relations in the same way as an ordinary algorithm operates on ordinary relations. Hence, we extend to symbolic relations the usual operators on relations.

Definition 4.1 Let N be a colored net, let M and M' be two symbolic relations of N. Then M.M' the product symbolic relation of M and M' is defined by : $M.M'(s', s) = \cup_{s'' \in S} M(s', s'') \circ M'(s'', s)$.

The product $R_M . R_{M'}$ of two relations R_M and $R_{M'}$ is usually defined by :

$$(s', c') \in R_M . R_{M'} (s, c) \Leftrightarrow \exists (s'', c'') \mid (s',c') \in R_M (s'', c'') \text{ and } (s'', c'') \in R_{M'} (s, c)$$

It is not difficult to show that the relation associated with the product of two symbolic relations is the product of the relations, i.e.,

$$R_{M.M'} = R_M . R_{M'}$$

<u>Definition 4.2</u> Let N be a colored net and M be a symbolic relation of N. M^n is recursively defined by M^0 = Identity and $M^n = M^{n-1} . M$. We define

$$M^+ = \bigcup_{n>0} M^n \qquad M^* = \bigcup_{n \in N} M^n$$

We could use R_{M^+} (resp. R_{M^*}) to compute the transitive (resp. reflexive transitive) closure of R_M. But as Warshall's algorithm is more efficient on ordinary graphs, we choose to extend this algorithm to symbolic relations. The first version that we propose for the extended algorithm is similar to the ordinary one except that we have substituted "∪" to "OR" and "o" to "AND", which are sound substitutions. However, the following algorithm is *a wrong version of the transitive closure*.

```
For s ∈ S do
     For s' ∈ S do
          For s" ∈ S do
               M(s", s') := M(s", s') ∪ M(s", s) o M(s, s') ;
```

If we apply this algorithm to our example, the result is not correct. In fact, we obtain :

$$SC(t_1, t_3) = SC(t_1, t_3) \cup SC(t_1, t_2) \circ SC(t_2, t_3) \quad = \overline{S - X}$$

meaning that an instance of t_3 indirectly conflicts with all the instances of t_1, except itself. But we have shown that actually (t_3, c) is in indirect structural conflict with (t_1, c). In fact, what is wrong with our algorithm is that we do not compute the transitive closure inside a node (i.e., a set of nodes in the unfolded net). Thus, a correct version of the algorithm would be :

```
For s ∈ S do M(s,s) := M(s,s)+
For s ∈ S do
     For s' ∈ S do
          For s" ∈ S do
               M(s", s') := M(s", s') ≈ M(s", s) o M(s, s)* o M(s, s')
```

Informally, the proof of the algorithm is the same as the one of the classical algorithm except that when we build new paths from the nodes of s' to the node of s" meeting s, these paths may meet s more than once (as s denotes a set of nodes in the unfolded net). In order to build the reflexive transitive closure, we only need to substitute $M(s, s)^*$ to $M(s, s)^+$ in the first line of the algorithm.

Once more, the efficiency of the method relies on a cheap computation of the operations involved in the algorithm. The following tables prove that the transitive and the reflexive transitive closure are easily obtained for the basic functions of URNs.

+	$\overline{0}$	$\overline{X}$	$\overline{S}$	$\overline{S - X}$
	$\overline{0}$	$\overline{X}$	$\overline{S}$	$\overline{S}$ °

° only if | C | > 1 . If | C | = 1, the result is $\overline{0}$

*	$\overline{0}$	$\overline{X}$	$\overline{S}$	$\overline{S - X}$
	$\overline{X}$	$\overline{X}$	$\overline{S}$	$\overline{S}$ ††

†† only if | C | > 1. If | C | = 1, the result is $\overline{X}$

Hence, if we consider a set of symbolic functions stable by the operations { ∪ , o , *, +}, the complexity of the (reflexive) transitive closure by the symbolic algorithm is O(n^3).

5. CONCLUSIONS

We have presented in this paper a general method to extend the structural analysis results from ordinary Petri nets to colored nets. This method is based on the definition of symbolic relations, which are in fact "folded" relations between the nodes of the unfolded Petri net associated to the colored net.

The advantage of this approach is that if the color functions used in a net fulfil some stability properties, the operations on the symbolic relations do not cost more than the equivalent operations on ordinary relations. Moreover, the results obtained with symbolic relations are valid for a family of nets with the same structure and different cardinalities for the color domains. Thus, even if some discussion on the values of these cardinalilties may be necessary, the algorithms that use symbolic relations work in a parametric way. This assertion has been illustrated by the presentation of the transitive closure algorithm.

We have shown in this paper how to compute some structural properties for unary regular nets using symbolic relations. And we strongly believe that this process can be applied to extended classes of models, provided that the functions of the model verify some stability conditions. Such a class of nets could be Well-Formed Nets [Chi 91] which have the same expressive power as colored nets.

6. REFERENCES

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