A Symbolic Reachability Graph for Coloured Petri Nets

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Abstract

Coloured Petri nets are well suited to the modelling of symmetric systems. Model symmetries can be usefully exploited for the sake of analysis efficiency as well as for modelling convenience. We present a reduced reachability graph called *symbolic reachability graph* that enjoys the following properties: 1) it can be constructed directly by an efficient algorithm without considering the actual state space of the model 2) it can be substantially smaller than the ordinary reachability graph 3) its analysis provides equivalent results as the analysis of the ordinary reachability graph. The construction procedure for the symbolic reachability graph is completely effective in the case of a syntactically restricted class of coloured nets called "well-formed nets", while for the unrestricted case of coloured nets some procedures may not be easily implementable in algorithmic form.

1 Introduction

Ordinary Petri nets [1, 2] are a good modelling tool for a precise representation of concurrent asynchronous systems of moderate size. Their terse graphic representation and their sound mathematical semantics allow a clear understanding of complex behavioural phenomena such as concurrency, conflict, synchronization, etc. A natural extension of the Petri net formalism to allow the representation of larger systems is the introduction of "Colour" structures to identify tokens. Coloured (or in general High-level) Petri nets (CPNs) [3, 4] allow a concise graphical representation of large *symmetric* systems made up of the repetition of several instances of some basic net structures.

The use of High-level Petri nets becomes particularly effective in practical application when the complexity of the analysis of coloured models depends on the basic structure of the model but not on the cardinalities of the colour sets. If this is the case, the verification of interesting model properties can be parametric in the actual colour definitions, thus yielding results that are valid for classes of models instead of a single model. For example, in some cases, invariant analysis may be parametric [5]. Unfortunately, few behavioural properties of a coloured Petri net model can be verified using parametric analysis techniques. In [6] an example of proof of correctness for a CPN model of a concurrent algorithm is shown that is parametric on the number of processes that execute the algorithm. Most of the interesting behavioural properties of a CPN model in general can be easily studied only by computing the reachability graph of the net, whose size depends on the cardinalities of the colour sets.

Even though an actual parametrization of the reachability graph analysis appears to be very difficult to obtain, one can nevertheless try and optimize the construction and analysis of the reachability graph of a CPN model by exploiting the symmetries that are inherent to a good exploitation of the CPN modelling formalism.

Aiming at reducing the size of the graph to analyze, Huber et al. [7] proposed to group some markings into equivalence classes. The construction of such classes is based on the (non-automatic) definition of behavioural symmetries, from which an equivalence relation is deduced that is used as a grouping criterion. For "Regular Nets" (a subclass of CPNs) Haddad [8] proposed another reduced graph, the Symbolic Reachability Graph (SRG). Besides classes of markings, classes of firings are created during the construction of the SRG. Another method is proposed for safe Predicate/Transition nets in [9]. This method is based on the binding of a variable (instead of a constant) colour when firing a transition. The variables then appear in parametrized markings. However, as symmetries are not taken into account, these variables may denote colours with potentially different behaviours. Hence, the graph obtained is more compact than the previous ones but even for the proof of basic properties such as deadlocks, a partial implicit unfolding of the parametrized graph is necessary.

From each of the two first methods, we can extract a key idea. By studying the reachability tree algorithm proposed by Huber et al., we notice that it is possible to define a reduced graph for any coloured net, even if all the procedures that we use in the construction of this graph are not algorithmic. Compared with this method, the main improvements of the symbolic reachability graph are twofold: first, the construction of equivalence classes of firing; second, the definition of a unique (or canonical) representative for each class of markings and each class of firings. The SRG is thus usually smaller than the reachability tree proposed in [7].

In this paper we extend the notion of SRG to the general case of CPNs. In the particular case of Well-formed coloured nets [10] (a CPN model in which the syntax for the definition of colour classes and functions is formally restricted to linear composition of a few basic functions) we have already shown an effective algorithm for the generation of the SRG. In this paper we formalize the notion of SRG even for the cases in which no effective algorithms may be found to construct them, and show how the SRG can be used instead of the actual RG to compute interesting model behavioural properties. All formal results are applied to the classic CPN model of the five philosophers problem in order to exemplify them.

The balance of the paper is as follows. Section 2 contains a definition of CPNs and of their basic symmetry properties. Section 3 provides an informal explanation of the SRG analysis technique. Section 4 presents the formal definition of SRG for general CPNs and outlines a (non-effective) construction algorithm. Section 5 defines the properties of the SRG of a CPN and proves their relation with the behavioural properties of the CPN model. Section 6 contains concluding remarks and perspectives of the work.

2 Coloured nets and Symmetries

A coloured Petri net is a net in which tokens are identified by colours. Colour domains are associated with places and transitions and determine which colours can mark the place (resp. fire the transition). When firing a transition, a number of tokens is taken from each input place, according to the incidence function labelling the arc between the place and the transition. In this paper, we will not consider the case of inhibitor arcs, neither transitions with priorities. Anyway the results obtained, being based on an interleaving semantics of bounded nets, can be directly extended to similar nets with priorities and inhibitor arcs.

Definition 2.1 *A (finite) multiset a on a finite non-empty set A is a function $a \in [A \rightarrow I\!N]$. A multiset a on a finite set A is called finite multiset.*

We will use $Bag(A)$ to denote the set of finite multisets on A. Intuitively, a multiset is a set that can contain several occurrences of the same element. It can be represented as a formal sum :

$$a = \sum_{x \in A} a(x).x$$

in which the non-negative integer $a(x)$ gives the number of occurrences of the element x in the multiset a. Thus, for two multisets a and b on A, we have :

$$a \leq b \iff \forall x \in A,\ a(x) \leq b(x)$$

We can also define the sum of two multisets a and b:

$$a + b = \sum_{x \in A} [a(x) + b(x)].x$$

or the difference, for $a \geq b$:

$$a - b = \sum_{x \in A} [a(x) - b(x)].x$$

A linear application on $Bag(A)$ will be defined by :

$$\forall\ a, b \in Bag(A), \qquad f(a + b) = f(a) + f(b)$$

2.1 Coloured net

We recall here the formal definition of a coloured Petri net. As shown in [12], this definition, although syntactically different, is equivalent to the one in [3].

Definition 2.2 *A coloured Petri net is a 6-tuple* $N = \langle P, T, C, W^-, W^+, M_0 \rangle$ *where*

P *is a set of places*

T *is a set of transitions, verifying* $P \cap T = \emptyset$, $P \cup T \neq \emptyset$

C *is the colour function, defined from* $P \cup T$ *into a set of finite non-empty sets called colour domains,*

W^-, W^+ *are the input and output functions, defined on* $P \times T$*, such that* $W^-(p,t)$ *and* $W^+(p,t)$ *belong to the set of linear applications mapping* $Bag(C(t))$ *onto* $Bag(C(p))$*, for all* $(p,t) \in P \times T$*,*

M_0 *the initial marking is a function defined on* P*, such that* $M_0(p) \in Bag(C(p))$*, for all* $p \in P$*.*

Although the input and output functions are defined on $Bag(C(t))$, we can limit the definition of their values to elements of $C(t)$ only. The values for the elements of $Bag(C(t))$ can be obtained using the linearity of the applications.

Definition 2.3 *Firing rule. A transition* t *is enabled for colour* c *in marking* M *(denoted by* $M[t,c\rangle$*) iff*

$$\forall p \in P,\ M(p) \geq W^-(p,t)(c)$$

The marking M' *obtained after the firing of* (t,c) *is computed as :*

$$\forall p \in P,\ M'(p) = M(p) + W^+(p,t)(c) - W^-(p,t)(c)$$

We will use the notation $M[t,c\rangle M'$ *to indicate this reachability relation*

Using the firing rule, it is possible to construct a reachability graph, whose nodes are the markings obtained from the initial marking by firing one or more transitions. An arc between two markings is labeled by the name of the transition and the colour whose firing determines the marking change.

Example Throughout the paper, we will consider the well-known synchronization problem of the dining philosophers. This situation is modeled by the coloured Petri net in Figure 1. A set of philosophers spend their lives thinking and eating. They share a common circular table laid with forks, one for each philosopher. From time to time a philosopher gets hungry and tries to pick up both his fork and that of his left-hand neighbour ($Take$). Thus, if at least one of his neighbours is eating he must wait until both neighbours have finished. Once the philosopher has finished eating he puts down both forks (Put) and resumes thinking again. This process is repeated indefinitely.

In the initial marking it is possible to fire transition $Take$ for any philosopher. Let us choose ph_4 arbitrarily; then we obtain :

$M_0(Thinking) = ph_0 + ph_1 + ph_2 + ph_3 + ph_4$
$M_0(Forks) = f_0 + f_1 + f_2 + f_3 + f_4$ $\qquad M_0(Eating) = 0$
The incidence functions around transition $Take$ are null for ph_4, except :
$W^-(Thinking,\ Take)(ph_4) = ph_4$
$W^-(Forks,\ Take)(ph_4) = f_4 + f_0$ $\qquad W^+(Eating,\ Take)(ph_4) = ph_4$
Hence, the marking obtained after firing $(Take, ph_4)$ is
$M'(Thinking) = ph_0 + ph_1 + ph_2 + ph_3$ $\qquad M'(Forks) = f_1 + f_2 + f_3$ $\qquad M'(Eating) = ph_4$

2.2 Symmetries

Coloured nets are particularly well suited to represent systems that have some behavioural symmetry properties. If we consider our example, firing transition $Take$ for philosopher $ph4$ or for philosopher $ph2$ will lead to very similar states. Actually, the two states obtained after firing are identical within a rotation. Moreover, they allow transition firings that are also identical within a rotation. Thus, we may consider these two states as *symmetric*. The notion of symmetry is not quite simple, as it is related to transition firings, hence to incidence functions. In the following we shall start by assuming that the modeller is able to define a group S of behavioural symmetries on the model, and that these symmetries verify some properties. Later on we shall overcome this assumption. These properties allow the modeller to verify that his set of symmetries is correct, i.e., that two states equal within a symmetry [1] have the same behaviour. However these properties are not constructive, so that they do not help in the identification of potential symmetries.

We start by recalling the notion of group operating on a set. Subsequently we introduce the definition of symmetries and the notion of admissible symmetry. Finally, we prove that the application of a permutation to a marking and to the colour instances of a transition preserves the firing relation.

2.2.1 Group operating on a set

In order to study the effect of a symmetry on a marking, we recall the definition of a group and the notion of a group operating on a set, which is a classical algebra notion. We will use this definition to study the operations of a group of symmetries not only on the markings, but also on the colours and the incidence functions.

Definition 2.4 $(G, \circ)$ *is a group iff the following properties are fulfilled :*

- $\forall x, y \in G, \quad x \circ y \in G$
- $\exists e \in G,\ \forall x \in G, \quad x \circ e = e \circ x = x$
- $\forall x \in G,\ \exists x^{-1} \in G, \quad x \circ x^{-1} = x^{-1} \circ x = e$

[1] that is, there exists a symmetry s such that $M_1 = s.M_2$

- $\forall x, y, z \in G, \quad x \circ (y \circ z) = (x \circ y) \circ z$

Definition 2.5 *The operation on the left (resp. on the right) of a group* $(G, \circ)$ *on a set* E *is a mapping* $G \times E \to E$ *(resp.* $E \times G \to E$*) such that, if we denote by* $g.x$ *the image of* (g, x)*,* $g \in G$*,* $x \in E$*, we have :*

- $\forall g, g' \in G, \ \forall x \in E, \quad (g \circ g').x = g.(g'.x) \quad (resp.\ x.(g \circ g') = (x.g).g')$
- $\forall x \in E, \quad e.x = x$ *where* e *is the neutral element of the group.*

Definition 2.6 *Let* G *be a group operating on* E*. The relation*

$$\exists g \in G, \qquad y = g.x$$

is an equivalence relation on E*. The equivalence class of* x *in* E *is called orbit of* x *and denoted by* $orb(x)$*. The elements of* G *that let* x *invariant form the isotropy subgroup* G_x *of* x *:*

$$G_x = \{g \in G \mid g.x = x\}$$

2.2.2 Symmetry and Equivalence

The use of groups of symmetries for the determination of equivalent markings has been first introduced in [12].

Definition 2.7 *A symmetry* s_C *on a colour domain* C *is a permutation on* C*. A symmetry* s *on a net is a family of symmetries* s_C *indexed by the set* $\mathcal{C} = \{C(r) | r \in P \cup T\}$ *of the colour domains that appear in the net.*

We denote by ξ the set of symmetries on a net. It directly comes from the properties of permutations that $(\xi, \circ)$ is a group. Actually, a set of permutations $\{s_C\}$ is associated with every colour domain C of the net. The composition of two permutations on C is still a permutation on C. The identity function on C is a permutation on C and also the neutral element for composition. Every permutation s_C has a symmetric element s_C^{-1}, which is also a permutation on C. Finally, the composition of permutations is associative. As a symmetry on a net is a family of permutations indexed by the set of colour domains of the net, we can conclude that symmetries verify the same properties as permutations, and hence $(\xi, \circ)$ is a group.

As $(\xi, \circ)$ is a group, we now examine the different sets on which it can operate. We will illustrate these operations by considering a symmetry $s = (s_1, s_2)$ on the model of the philosophers. We choose :

$s_1(ph_0) = ph_2$	$s_1(ph_1) = ph_0$	$s_1(ph_2) = ph_3$	$s_1(ph_3) = ph_4$	$s_1(ph_4) = ph_1$
$s_2(f_0) = f_4$	$s_2(f_1) = f_2$	$s_2(f_2) = f_0$	$s_2(f_3) = f_1$	$s_2(f_4) = f_3$

We can define the operations of $(\xi, \circ)$ on :

- colour domains :
$$s.c = s_C(c), \quad c \in C$$
for instance, $s.ph_2 = ph_3$ or $s.f_0 = f_4$.
- multisets of colours :
$$s.\left(\sum_{c \in C} \lambda_c.c\right) = \sum_{c \in C} \lambda_c.s.c, \quad c \in C$$
In our example, $s.(ph_1 + 2.ph_3) = ph_0 + 2.ph_4$.
Notice that for two multisets a and b on $Bag(C)$, we have : $s.(a + b) = s.a + s.b$, and if $a \leq b$, then $s.a \leq s.b$ and $s.(b - a) = s.b - s.a$.

- markings :

$$s.M : \quad (s.M)(p) = s.(M(p))$$

Applying this definition to the marking presented in Section 2.1 we have

$M(Thinking) = ph_0 + ph_1 + ph_2 + ph_3$ $\quad M(Forks) = f_1 + f_2 + f_3$ $\quad M(Eating) = ph_4$
$s.M(Thinking) = ph_0 + ph_2 + ph_3 + ph_4$ $\quad s.M(Forks) = f_0 + f_1 + f_2$ $\quad s.M(Eating) = ph_1$

As $M(p)$ is an element of $Bag(C(p))$, this operation is a particular case of the operation defined above, and thus we have : $\forall c \in C(p), \quad (s.M)(p)(s.c) = M(p)(c)$
e.g., $M(Eating)(ph_4) = 1$ and $s.M(Eating)(ph_1) = 1$

- incidence functions :
  - on the left :

$$s.W^*(p,t) : \quad s.W^*(p,t)(c) = s_{C(p)}.(W^*(p,t)(c))$$

where $W^*(p,t)$ is a macro-notation that can be bound to $W^-(p,t)$ or $W^+(p,t)$. As above, $s.W^*(p,t)(c)$ is defined on $Bag(C(p))$, and thus we have :

$$\forall d \in C(p), \quad [s.W^*(p,t)](c)(s.d) = W^*(p,t)(c)(d)$$

  - on the right :

$$W^*(p,t).s : \quad (W^*(p,t).s)(c) = W^*(p,t)(s_{C(t)}(c))$$

i.e.,

$$\forall d \in C(p), \quad [W^*(p,t).s](c)(d) = W^*(p,t)(s.c)(d)$$

For instance, we have $W^-(Forks, Take)(ph_4) = f_0 + f_4$. The operation of s on the left and on the right respectively gives
$s.W^-(Forks, Take)(ph_4) = f_3 + f_4$ and $W^-(Forks, Take).s(ph_4) = f_1 + f_2$

The construction of the symbolic reachability graph relies on the definition of a set of admissible symmetries that are used for the construction of equivalence classes of places and firings. Two approaches are possible for the definition of the set of symmetries. They can be either explicitely described by the modeller, and in that case the algorithm for the construction of the SRG must check that they are correct, i.e., that they fulfil the suitable properties. Or they can be automatically determined by the algorithm. We prefer the second solution because on the one hand the determination of the symmetries may be complex for general coloured nets, and on the other hand, it provides a completely automatic construction of the symbolic reachability graph. An algorithm to compute the generators of the symmetry group can be found in [11].

Definition 2.8 *The set S of admissible symmetries is a subset of the set ξ of symmetries that satisfies the following conditions :*

1. $(S, \circ)$ *is a subgroup of* $(\xi, \circ)$
2. $\forall s \in S,\ \forall p \in P,\ \forall t \in T, \quad s.W^*(p,t) = W^*(p,t).s$, *i.e.,*
   - $s.W^+(p,t) = W^+(p,t).s$
   - $s.W^-(p,t) = W^-(p,t).s$

The aim of the second condition of the definition is to ensure that two markings equal within an element $s \in S$ allow transition firings that are also equal within the application of s to the colour instances.

Example In the model of the dining philosophers, all the places and transitions have the same colour domain C_1, except place $Fork$ that has colour domain C_2. The set of symmetries of the net is thus the set of functions (s_1, s_2), where s_1 is a permutation on C_1 and s_2 is a permutation on C_2. Among these symmetries, only those that verify the second condition of Definition 2.8, i.e., that commute with the colour functions of the net, are admissible. Without entering the details, because of the structure of function $Need$, only the functions (s_1, s_2) such that s_1 and s_2 modify the indices in the same way, i.e., if $s_1(ph_i) = ph_j$ then $s_2(f_i) = f_j$, and such that s_1 (and also s_2) is a rotation are admissible. The reason is that function $Need$ is the sum of an identity function and a rotation applied to a fork, and only rotations commute with rotations.

Property 2.1 *The application of an admissible symmetry to a marking and to the colour instance of a transition preserves the firing :*

$$\forall t \in T,\ \forall c \in C(t),\ \forall s \in S, \qquad M[t,c\rangle M' \iff s.M[t,s.c\rangle s.M'$$

For instance, on the graph of Figure 2, by applying the admissible symmetry $s = (s_1, s_2)$, with $s_1(ph_i) = ph_{(i+1) mod\ 5}$ and $s_2(f_i) = f_{(i+1) mod\ 5}$, to the firing $M_0[Take, ph_0\rangle M_1$, we obtain the firing $M_0[Take, ph_1\rangle M_2$.

Proof Enabling : We first show that $M[t,c\rangle \iff s.M[t,s.c\rangle$.
$M[t,c\rangle \iff \forall p \in P,\ \ M(p) \geq W^-(p,t)(c)$.
As $M(p)$ and $W^-(p,t)(c)$ are both elements of $Bag(C(p))$, if we apply the former remark about the operation of a symmetry on a multiset, we obtain : $\forall p \in P,\ \ s.M(p) \geq s.W^-(p,t)(c)$
$\iff \forall p \in P,\ \ s.M(p) \geq W^-(p,t).s(c)$ (because s is admissible).
Using the definition of the operation of s on $W^*(p,t)$, we have $W^-(p,t).s(c) = W^-(p,t)(s.c)$. Hence, $s.M[t,s.c\rangle$
Firing : $M[t,c\rangle M' \iff \forall p \in P,\ M'(p) = M(p) + W^+(p,t)(c) - W^-(p,t)(c)$
$\iff s.M'(p) = s.(M(p) + W^+(p,t)(c) - W^-(p,t)(c))$.
Still using the remark about the operation of a symmetry on a multiset, we have :
$s.M'(p) = s.M(p) + s.W^+(p,t)(c) - s.W^-(p,t)(c)$. As s is admissible,
$s.M'(p) = s.M(p) + W^+(p,t)(s.c) - W^-(p,t)(s.c)$. Finally, $s.M[t,s.c\rangle s.M'$

As a consequence, two markings that can be obtained one from the other by applying a symmetry s enable the same transitions, except for the binding of these transitions. Thus, we can use the equivalence relation associated with the orbits of markings to define classes of markings.

Definition 2.9 *Two markings are equivalent iff*

$$\exists s \in S, \quad M' = s.M$$

3 Presentation and Discussion

Remark In the rest of the paper, we consider only coloured nets yielding a finite number of reachable markings, since we propose the complete enumeration of the RG as an analysis tool.

We briefly recall the construction of Huber's et al. reachability tree in order to present and discuss the improvements obtained by our algorithm. Actually, Huber's et al. algorithm is very close to the construction of an ordinary reachability graph. We will denote by SRG the graph obtained with this algorithm as well. In the algorithm that we present, the test for the equivalence of markings is performed by an exhaustive search of the existence of a symmetry mapping one marking onto the other. In many practical cases, the modeller may know more efficient ways to test for equivalence than the exhaustive search. Based on this knowledge, the efficiency of the algorithm may be improved ; however, the generality of the algorithm is lost in this case. The problem may be solved retaining the generality by using a canonical representative as we shall see in the following.

Huber's et al. algorithm

```
SRG := {M_0}
Push M_0
While not Empty(stack) do
    Pop M
    /* Firing test of all instance of all transitions */
    for all t ∈ T do
        for all c ∈ C(t) do
            if M[t,c⟩M' then
                /* Equivalence test for M' */
                New := True;
                for all s ∈ S do
                    M'' := s.M';
                    if M'' ∈ SRG then
                        New := False;
                        Goto Cont;
                    endif
                endfor /* End of equivalence test */
Cont :          if New then
                    Push M'
                endif
                SRG := SRG ∪{M[t,c⟩M'}
            endif
        endfor
    endfor
```

Let us emphasize that the difference from the ordinary reachability graph construction is the substitution of an equivalence test to the belonging test for M'. We can estimate the cost of this equivalence test, in the worst case where no efficient test method is provided together with the model: $|S|.O(\text{application of a symmetry}) + |S|.|SRG|.O(\text{test of equality})$. For instance, in the subgraph shown in Figure 2, one may need five applications of symmetries and ten tests of equality to find that M_2 is equivalent to M_1.

Here our first improvement comes into play. Let us assume that a representative is given for each equivalence class and that the computation of the representative of any marking can be obtained in a time of the same order of magnitude as for the application of a symmetry. We can transform the equivalence test by first computing the representative of the marking and then test the equality with each marking of the SRG. The cost is expressed by : $O(\text{application of a symmetry}) + |SRG|.O(\text{test of equality})$. In several practical cases, the same improvement can be easily obtained using Huber's et al. algorithm by adding a model specific test function. However, our proposed method relieves the modeller from this burden. If we compare with the first formula, we have divided the cost by $|S|$. We illustrate this improvement in the subgraph shown in Figure 3.

Let us look now at the initial marking of the net given in the example of section 2. Since all philosophers are in the same state, i.e., they are all thinking, if a transition is enabled for one philosopher it is enabled for any philosopher. Thus we could test each transition for one philosopher only and apply the symmetries to find the other possible firings. This technique is illustrated in Figure 4.

In a more general case we can still use this technique with an appropriate subset of symmetries deduced from each marking. If this determination is computationally cheap, then we can decrease the cost of testing by substituting some firing tests with the application of a symmetry. Indeed the firing test involves many computation steps while applying a symmetry is a single-step computation.

A deeper study (see the next section) shows that no information is lost if one generates only the representatives of markings and firing instances. For our example this gives the simplified subgraph shown in Figure 5. The next section will develop all these points in a more formal way.

4 Construction of the Symbolic Reachability Graph

4.1 Classes of Markings

The factorisation of markings in the symbolic reachability graph consists in grouping states into classes, and including in the graph only one representative for each class. While using the same basic principle as Huber's et al. that equivalent markings must allow equivalent firings, our algorithm produces a more compact graph. We can then develop a reachability subgraph from the representatives of classes only, without loss of information.

As markings are grouped into equivalence classes, instead of representing all of them in the graph, we define a representative for each class. Only the representative marking of each class is included in the graph. The choice of the representatives is completely arbitrary, and for the moment we do not suggest any particular solution to perform this choice.

Definition 4.1 *Let $\overline{M}$ be the representative of M. s_M is a symmetry such that $s_M.M = \overline{M}$ (there may be several symmetries that satisfy this relation).*

Notation We denote by $\overline{M}, \overline{M}'$ two different marking representatives, whereas $M, M' \in \overline{M}$ will denote two equivalent markings represented by $\overline{M}$.

The construction of equivalence classes of markings is an idea that already appears in [7]. On the contrary, the factorisation of firings that we present now is an original idea of the symbolic reachability graph [8], that has been independently studied also in [13] where it was called self-symmetry. Our aim is to be able to test all the possible firings from any marking in a class by studying only some of the possible firings from the representative of the class.

4.2 Firing Factorisation

Now that we have defined classes of markings, we want to define classes of firings in a similar way. Considering Property 2.1, we can notice that for any permutation s that leaves M invariant, if (t, c) is enabled then $(t, s.c)$ is also enabled. This is the key point for the definition of classes of firings. Actually, the isotropy subgroup of $\overline{M}$ defines equivalence classes of colours. Instead of testing the enabling of t for all colours of $C(t)$ we can test for only one colour in each equivalence class. The colour chosen for the test is again called representative. However, the marking obtained when firing the representative of a colour in the representative of a marking may not be a representative. As we want to construct a reachability graph including the representatives of markings, this fact must be taken into account in the definition of our symbolic firing rule.

Definition 4.2 *Let $S_M = \{s \in S | s.M = M\}$ be the isotropy subgroup of M. Let $C_M = C/S_M$ be the set of colour classes obtained when quotienting the colour domain $C \in \mathcal{C}$ by the group S_M. C_M is a set of equivalence classes, and for each class in C_M, we arbitrarily choose an element of the* *class as a representative. We define function α_M as the function which associates with any colour $c \in C$ the representative of the class of c in C_M.*

The next property immediately follows :

Property 4.1 *Let $\overline{c}$ be the representative of a class in C_M. Then for any colour c belonging to the class of $\overline{c}$, i.e., such that $\alpha_M(c) = \overline{c}$, there exists a symmetry $s \in S_M$ such that $s.c = \overline{c}$.*

Notice that the representative associated with a colour is local to a marking. As we want to define a SRG that includes only one representative for each class of markings, we define a symbolic firing rule based on the possibility of firing a transition for the representative of a colour in the representative of a marking. This symbolic firing rule allows us to build a SRG, and we will show in Section 5 that it is sound, i.e., that the main properties of the RG can be studied on our SRG.

In the following when no confusion may arise we will identify C_M with a reduced set containing only the representative of each class.

Definition 4.3 *The transition t can be symbolically fired in the marking $\overline{M}$ for the colour instance $\alpha_{\overline{M}}(c)$ representing c in $C_{\overline{M}}$, denoted by $\overline{M}[\![t, \alpha_{\overline{M}}(c)\rangle\!\rangle$, if and only if t can be fired in $\overline{M}$ for $\alpha_{\overline{M}}(c)$. The symbolic marking $\overline{M}'$ obtained after the symbolic firing is such that $\exists M'' \in \overline{M}'$ verifying $\overline{M}[t, \alpha_{\overline{M}}(c)\rangle M''$. Thus we have :*

$$\overline{M}[\![t, \alpha_{\overline{M}}(c)\rangle\!\rangle \overline{M}' \iff \exists M'' \text{ such that } \overline{M''} = \overline{M}' \wedge \overline{M}[t, \alpha_{\overline{M}}(c)\rangle M''.$$

4.3 General Algorithm for the Construction of the SRG

The advantage of the symbolic firing is that it allows us to construct a reduced reachability graph automatically, containing a minimal number of arcs and nodes. We outline here an algorithm for the construction of the SRG.

Recall the main points on which the construction of the SRG is based:

- a symbolic representative $\overline{M}$ is associated with each marking M
- the isotropy subgroup of $\overline{M}$ denoted $S_{\overline{M}}$ is associated with $\overline{M}$
- for each transition t, for all $c \in C(t)$, we choose a representative in $C(t)_{\overline{M}}$ that we denote $\overline{c}$.

```
M̄ := representative (M₀)
SRG := {M̄}
Push M̄
While not Empty(stack) do
      Pop M̄
      for all t ∈ T do
          for all c̄ ∈ C(t)_M̄ do
              if M̄[t, c̄⟩M' then
                 M̄' := representative (M')
                 if M̄' ∉ SRG then
                    Push M̄'
                 endif
                 SRG := SRG ∪M̄⟦t, c̄⟫M̄'
              endif
          end
      end
end
```

We apply this algorithm to the model of the philosophers. We choose as representative of a class of firings the element of the class with the minimum lexicographic value. We do the same for classes of markings, and we choose the following order of places to define the lexicographic value of a marking : $M = (M(Thinking),\ M(Forks),\ M(Eating))$.

In the model of the philosophers, the initial marking is symmetric, i.e., $\forall s \in S,\ s.M_0 = M_0$ and thus equal to its representative. Due to this symmetry, $S_{\overline{M_0}} = S$ and the firing need to be tested for only one color for each transition. Transition $Take$ is enabled, and its firing leads to a marking M_1. This marking is replaced by its representative $\overline{M_1}$, which is added to the SRG and represents the five markings that can be obtained from it by applying a rotation. When examining $\overline{M_1}$, it is clear that the only admissible symmetry that leaves it invariant is the identity. Actually, such a symmetry must be such that s_1 is a rotation that leaves ph_4 invariant, and we have seen also that the admissible symmetries in the model of the philosophers must be such that s_1 and s_2 modify the indices in the same way. As a consequence, the enabling test must be performed independently for every colour of each transition. We find that transition $Take$ is enabled for ph_1 and ph_2. The marking obtained after these firings have the same representative $\overline{M_2}$, and thus only one new marking is added to the SRG. In $\overline{M_1}$, transition Put is also enabled for ph_4 and returns to the initial marking. For $\overline{M_2}$ too, only the identity function leaves the marking invariant. Once again, the enabling test must be performed for every colour. The complete SRG for the model of the philosophers is given in Fig. 6.

Note that in the general case some of the procedures used by the algorithm cannot be implemented. This is the case for

- determining the symmetries of the model
- choosing the representative of a class of markings efficiently
- building classes of firings.

In order to overcome these problems we define a new class of coloured nets, the "*Well-formed Coloured Nets*." Because of the structure and the restricted syntax of this class, the procedures presented above can be implemented automatically. The complete process of the SRG construction for this class of nets was presented in [10]. We outline the way it is performed. In a Well-Formed Net, a colour domain is a Cartesian product of object classes. These classes group entities of the same kind, such as the class of forks or the class of philosophers. All objects within a class must have potentially the same behaviour, i.e., they must be able to perform the same actions at possibly different times. If not, the class must be divided in static subclasses, each of them including objects with homogeneous behaviour. A class C may be ordered. This is the case in our example, where the philosophers are ordered around the table in order to identify the right and left neighbours. As colour domains are defined by Cartesian products of object classes, the symmetries in Well-Formed Nets are obtained by composition of functions that apply to an object class. If the class is not ordered, the function may be any permutation, whereas for an ordered class the function must be a rotation. If the class is divided in static subclasses we have an additional restriction, namely: the image by the function of any object must belong to the same static subclass as the object. The symmetries of the model are defined implicitly and a-priori.

The representative of a marking is defined in terms of "*dynamic subclasses*." A dynamic subclass is a representation for a set of objects that have the same token distribution in the considered marking. This representation is not binded. All possible bindings of objects of the colour class in which a dynamic subclass is included yield the different ordinary markings that the symbolic marking represents.

The advantages of this representation are twofold. First, the equality of two symbolic representations is more efficient to test than the equivalence of two ordinary markings. Second (and perhaps more crucial), this representation can be used directly to implement a symbolic firing rule:

instead of binding transitions with objects we can bind them with dynamic subclasses. Hence after the firing we still obtain a *class* of markings that, after some automatic operation, is transformed into a representative.

Notice that the availability of a reduced graph is useful only if it can be used to prove directly the most important properties of coloured nets. We now present some properties that can be studied directly on the SRG.

5 SRG Properties

The properties we give in this section aim at establishing a correspondence between the SRG and the (ordinary) reachability graph of a CPN. We will illustrate the properties on the example of the philosophers, whose SRG is given in Figure 6 and whose reachability graph is given in Figure 7. The first properties that we present show how an ordinary firing is represented by a symbolic firing. We study in Sections 5.2 and 5.3 the correspondence between properties of the RG and properties of the SRG. The properties in the last section give the number of markings represented by a symbolic marking, and the number of outgoing arcs from one marking that are represented by a symbolic firing.

For the sake of simplicity, and without loss of generality, we consider here only the case where the initial marking of the net is symmetric, i.e., the application to the initial marking of any element in S leave this marking invariant. In this case, M_0 is the only element in its class and is equal to $\overline{M}_0$. If the initial state of the system is not symmetric, it is possible to add to the model an extra initialisation transition that will create a non-symmetric marking from a symmetric initial marking. Anyway, the extension of the properties and the proofs presented in this section to the case of a non-symmetric initial marking can be found in [14].

5.1 Basic Properties

Property 5.1 *Each ordinary firing is represented by a unique symbolic firing:*

$$M[t,c\rangle M' \implies \overline{M}[\![t,\overline{c}\rangle\!\rangle\overline{M}'$$

with $\overline{c} = \alpha_{\overline{M}}(s_M.c)$

Proof According to Property 2.1, $M[t,c\rangle M' \implies \overline{M}[t, s_M.c\rangle s_M.M'$
Let $\overline{c} = \alpha_{\overline{M}}(s_M.c)$ be the representative of $s_M.c$ in $\overline{M}$. By Property 4.1, $\exists s \in S_{\overline{M}}$ such that $s.(s_M.c) = \alpha_{\overline{M}}(s_M.c)$. Hence applying s to our relation we obtain $\overline{M}[t,\overline{c}\rangle(s \circ s_M).M'$
As $\overline{(s \circ s_M).M'} = \overline{M}'$, we finally obtain $\overline{M}[\![t,\overline{c}\rangle\!\rangle\overline{M}'$.

In our example, firing $M_1[Take, ph_3\rangle M_{13}$ can be mapped onto firing $M_4[Take, ph_1\rangle M_{41}$ by applying a rotation to the colors. As $M_4 = \overline{M_1}$ and M_{41} can be mapped onto $\overline{M_2}$, this firing is represented by the symbolic firing $\overline{M_1}[[Take, ph_1\rangle\rangle\overline{M_2}$.

Property 5.2 *A set of ordinary firings can be extracted from every symbolic firing such that the source markings belong to the class of the source symbolic marking :*

$$\overline{M}[\![t,\overline{c}\rangle\!\rangle\overline{M}' \implies \forall M \in \overline{M},\ \forall c' \text{ with } \alpha_{\overline{M}}(s_M.c') = \overline{c},\ \exists M'' \in \overline{M}' \text{ such that } M[t,c'\rangle M''$$

Proof $\overline{M}[\![t,\overline{c}\rangle\!\rangle\overline{M}' \implies \exists M_1$ such that $(\overline{M}[t,\overline{c}\rangle M_1 \ \wedge\ \overline{M}_1 = \overline{M}')$.
Let c' be such that $\alpha_{\overline{M}}(s_M.c') = \overline{c}$. Let $c'' = s_M.c'$.
As $\overline{c}$ is the representative of c'', $\exists s \in S_{\overline{M}}$ such that $s.c'' = \overline{c}$.

$S_{\overline{M}}$ is a group, hence $\exists s^{-1} \in S_{\overline{M}}$, and applying Property 2.1 we obtain $s^{-1}.\overline{M}[t, s^{-1}.\overline{c}\rangle s^{-1}.M_1$.
This can also be written as $\overline{M}[t, c''\rangle s^{-1}.M_1$. For all $M \in \overline{M}$ we can apply s_M^{-1} to this firing and obtain $M[t, s_M^{-1}.c''\rangle (s_M^{-1} \circ s^{-1}).M_1$.
As $\overline{(s_M^{-1} \circ s^{-1}).M_1} = \overline{M}'$, we finally obtain $M[t, c'\rangle M''$ where $M'' = (s_M^{-1} \circ s^{-1}).M_1$.

Property 5.3 *An ordinary firing can be extracted from every symbolic firing such that the destination marking belongs to the class of the destination symbolic marking:*

$$\overline{M}[\![t, \overline{c}\rangle\!\rangle \overline{M}' \implies \forall M_2 \in \overline{M}',\ \exists M_1 \in \overline{M},\ \exists c' \text{ such that } M_1[t, c'\rangle M_2.$$

Proof $\overline{M}[\![t, \overline{c}\rangle\!\rangle \overline{M}' \implies \exists M''$ such that $(\overline{M}[t, \overline{c}\rangle M'' \ \wedge\ \overline{M}'' = \overline{M}')$.
Applying $s_{M''}$ (the permutation that maps M'' on $\overline{M}'' = \overline{M}'$) to this firing we obtain another firing, namely $s_{M''}.\overline{M}[t, s_{M''}.\overline{c}\rangle \overline{M}'$.
However $\forall M_2 \in \overline{M}'$, $\exists s_{M_2} \in S$ such that $s_{M_2}.M_2 = \overline{M}'$. Hence applying $s_{M_2}^{-1}$ to the former relation, we obtain
$(s_{M_2}^{-1} \circ s_{M''}).\overline{M}[t, (s_{M_2}^{-1} \circ s_{M''}).\overline{c}\rangle M_2$.
This is the relation given in Property 5.3, with $M_1 = (s_{M_2}^{-1} \circ s_{M''}).\overline{M}$ and $c' = (s_{M_2}^{-1} \circ s_{M''}).\overline{c}$.

In our example, by applying an admissible permutation on $\overline{M_1}$, we obtain the set of markings $M_i, i = 1, \ldots, 5$. As there is no admissible permutation that leaves $\overline{M_1}$ unchanged, any firing from $\overline{M_1}$ is alone in its class, and thus, $\forall c,\ \alpha_{\overline{M}}(c) = c$. We can check on the SRG and RG that to the symbolic firing $\overline{M_1}[[Take, ph_1\rangle\rangle \overline{M_2}$ corresponds an ordinary firing $M_i[Take, ph_j\rangle M_{ij}$, for every $i = 1, \ldots, 5$ and with $j = (i+2) mod\ 5$. Thus, Property 5.2 is verified for this firing. For the same symbolic firing, as $M_{ij}, i = 1, \ldots, 5$ and $j = (i+2) mod\ 5$ is the set of markings represented by $\overline{M_2}$, Property 5.3 holds true too.

Property 5.3 can be considered to be weaker than Property 5.2, as it exhibits only one colour. However this is due to the definition we chose for the symbolic firing. Indeed a symbolic firing is a set of arcs departing from the same marking, but we could have chosen to group arcs that reach the same marking as well. In this case, Property 5.2 would have been weakened. We chose the solution that seemed the most intuitive to us, and the implication given in Property 5.3 is powerful enough to prove interesting results on the SRG.

The following three properties extend the previous properties to firing sequences.

Property 5.4 *Let* $\sigma = ((t_{u_1}, c_{u_1}),\ (t_{u_2}, c_{u_2}),\ \ldots,\ (t_{u_k}, c_{u_k})),\quad c_{u_i} \in C(t_{u_i})$, *be a firing sequence such that*

$$M_1[t_{u_1}, c_{u_1}\rangle\ M_2[t_{u_2}, c_{u_2}\rangle\ M_3\ \ldots\ M_{k-1}[t_{u_{k-1}}, c_{u_{k-1}}\rangle\ M_k$$

(which will be also denoted by $M_1[\sigma\rangle M_k$*).*
Then there exists a symbolic firing sequence $\overline{\sigma}$

$$\overline{M}_1[\![t_{u_1}, \overline{c}_{u_1}\rangle\!\rangle\ \overline{M}_2[\![t_{u_2}, \overline{c}_{u_2}\rangle\!\rangle\ \overline{M}_3\ \ldots\ \overline{M}_{k-1}[\![t_{u_{k-1}}, \overline{c}_{u_{k-1}}\rangle\!\rangle\ \overline{M}_k$$

such that $M_i \in \overline{M}_i$ *and* $\overline{c}_{u_i} = \alpha_{\overline{M}_i}(s_{M_i}.c_{u_i})$. *(We shall use the notation* $\overline{M}_1[\![\overline{\sigma}\rangle\!\rangle \overline{M}_k$*)*

Proof By induction on the length of the sequence: in case the sequence is empty, the property holds true trivially; the induction step follows immediately by Property 5.1.

Property 5.5 *Let* $\overline{\sigma} = ((t_{u_1}, \overline{c}_{u_1}),\ (t_{u_2}, \overline{c}_{u_2}),\ \ldots,\ (t_{u_k}, \overline{c}_{u_k})),\quad \overline{c}_{u_i} \in C(t_{u_i})$, *be a symbolic firing sequence such that*

$$\overline{M}_1[\![t_{u_1}, \overline{c}_{u_1}\rangle\!\rangle\ \overline{M}_2[\![t_{u_2}, \overline{c}_{u_2}\rangle\!\rangle\ \overline{M}_3\ \ldots\ \overline{M}_{k-1}[\![t_{u_{k-1}}, \overline{c}_{u_{k-1}}\rangle\!\rangle\ \overline{M}_k$$

Then $\forall M'_1 \in \overline{M}_1,\ \exists M'_2 \in \overline{M}_2,\ \ldots,\ M'_k \in \overline{M}_k,\ \exists c'_{u_i}$ *with* $\alpha_{\overline{M}_i}(s_{M_i}.c'_{u_i}) = \overline{c}_{u_i}$ *such that*

$$M'_1[t_{u_1}, c'_{u_1}\rangle\ M'_2[t_{u_2}, c'_{u_2}\rangle\ M'_3\ \ldots\ M'_{k-1}[t_{u_{k-1}}, c'_{u_{k-1}}\rangle\ M'_k$$

Proof By induction on the length of the sequence: in case the sequence is empty, the property holds true trivially; the induction step follows immediately by Property 5.2.

Property 5.6 *Let* $\overline{\sigma} = ((t_{u_1}, \overline{c}_{u_1}),\ (t_{u_2}, \overline{c}_{u_2}),\ \ldots,\ (t_{u_k}, \overline{c}_{u_k})),\quad \overline{c}_{u_i} \in C(t_{u_i})$, *be a symbolic firing sequence such that*

$$\overline{M}_1[\![t_{u_1}, \overline{c}_{u_1}\rangle\!\rangle\ \overline{M}_2[\![t_{u_2}, \overline{c}_{u_2}\rangle\!\rangle\ \overline{M}_3\ \ldots\ \overline{M}_{k-1}[\![t_{u_{k-1}}, \overline{c}_{u_{k-1}}\rangle\!\rangle\ \overline{M}_k$$

Then $\forall M'_k \in \overline{M}_k,\ \exists M'_1 \in \overline{M}_1,\ \ldots,\ M'_{k-1} \in \overline{M}_{k-1},\ \exists c'_{u_1},\ \ldots,\ c'_{u_k}$ *such that*

$$M'_1[t_{u_1}, c'_{u_1}\rangle\ M'_2[t_{u_2}, c'_{u_2}\rangle\ M'_3\ \ldots\ M'_{k-1}[t_{u_{k-1}}, c'_{u_{k-1}}\rangle\ M'_k$$

Proof By induction on the length of the sequence: in case the sequence is empty, the property holds true trivially; the induction step follows immediately by Property 5.3.

5.2 Structural Properties

The two following properties compare the reachability in the SRG to the ordinary reachability. The first one compares the symbolic and ordinary reachability sets whereas the second one concerns the finiteness of the graph.

Property 5.7 *Reachability equivalence. Let* $[\![\overline{M}_0\rangle\!\rangle$ *be the symbolic reachability graph obtained from the initial symbolic marking* $\overline{M}_0$. *Let* $[M_0\rangle$ *be the reachability graph obtained from the initial marking* M_0. *Then we have the following property :*

$$\{M \mid M \in [M_0\rangle\} = \{M \mid \overline{M} \in [\![\overline{M}_0\rangle\!\rangle\}$$

Proof We prove this property by showing the double inclusion.

$\subseteq$: $M \in [M_0\rangle \iff \exists\sigma$ a firing sequence such that $M_0[\sigma\rangle M$. Thus according to Property 5.4, $\exists\overline{\sigma}$ a symbolic firing sequence such that $\overline{M}_0[\![\overline{\sigma}\rangle\!\rangle\overline{M}$. As a consequence $\overline{M} \in [\![\overline{M}_0\rangle\!\rangle$.

$\supseteq$: $\overline{M} \in [\![\overline{M}_0\rangle\!\rangle \iff \exists\overline{\sigma}$ a symbolic firing sequence such that $\overline{M}_0[\![\overline{\sigma}\rangle\!\rangle\overline{M}$. Thus from Property 5.6 $\forall M' \in \overline{M},\ \exists\sigma'$ such that $M_0[\sigma'\rangle M'$. Hence, $M' \in [M_0\rangle$.

In the example of the philosophers, every marking obtained by applying an admissible permutation on a marking of the SRG belongs to the RG. Vice-versa, there is no marking in the RG that cannot be mapped onto a marking of the SRG by applying an admissible permutation.

Property 5.8 *The two following properties are equivalent (remember that only finite colour sets are considered).*

i) $[M_0\rangle$ *is infinite.*

ii) $[\![\overline{M}_0\rangle\!\rangle$ *is infinite.*

Proof

i) $\Longrightarrow$ ii) : $[M_0\rangle$ infinite $\Longrightarrow$ $\exists p \in P$, $\exists c \in C(p)$, $\forall B \in I\!N$, $\exists M \in [M_0\rangle$ such that $M(p)(c) \geq B$. Using the symmetry s_M that maps M on $\overline{M}$, we have by definition $M(p)(c) = \overline{M}(p)(s_M.c)$. Thus, $\overline{M}(p)(s_M.c) \geq B$. Knowing from Property 5.7 that $M \in [M_0\rangle \Longrightarrow \overline{M} \in [\![\overline{M}_0\rangle\!\rangle$, we see that $[\![\overline{M}_0\rangle\!\rangle$ is unbounded, hence infinite.

ii) $\Longrightarrow$ i) : we show that $\neg$ i) $\Longrightarrow$ $\neg$ ii).
Assume that $[M_0\rangle$ is finite.
Thus $\exists B \in I\!N$ such that $\forall p \in P$, $\forall c \in C(p)$, $\forall M \in [M_0\rangle$, $M(p)(c) < B$.
Let $\overline{M}$ be any marking in $[\![\overline{M}_0\rangle\!\rangle$. $\exists \overline{\sigma}$ a firing sequence such that $\overline{M}_0[\![\overline{\sigma}\rangle\!\rangle\overline{M}$.
From Property 5.5, we know that $\exists M' \in \overline{M}$, $\exists \sigma'$, $M_0[\sigma'\rangle M'$.
As $M' \in [M_0\rangle$, we have $\forall p \in P$, $\forall c \in C(p)$, $M'(p)(c) < B$. Also, $M' \in \overline{M}$, hence $\exists s$ a symmetry such that $s.M' = \overline{M}$. As a consequence, $M'(p)(c) < B \Longrightarrow \overline{M}(p)(s.c) < B$.
This is true for any $c \in C(p)$, and s defines a bijection among the colours of $C(p)$. Hence the SRG is bounded and finite.

5.2.1 Strong connection

When studying a Petri net model it is important to be able to determine whether the corresponding reachability graph is strongly connected. This is especially the case for stochastic Petri nets where the strong connection of the reachability graph is directly related to the notion of model ergodicity. The property given in this section links the strong connection of the SRG to that of the corresponding ordinary reachability graph.

Property 5.9 *The two following properties are equivalent. They relate the strong connection of the SRG to that of the ordinary reachability graph it represents.*

i) $[M_0\rangle$ *is strongly connected,*

ii) $[\![\overline{M}_0\rangle\!\rangle$ *is strongly connected.*

Proof

i) $\Longrightarrow$ ii) : Let $\overline{M} \in [\![\overline{M}_0\rangle\!\rangle$. $\exists \overline{\sigma}$ a firing sequence such that $\overline{M}_0[\![\overline{\sigma}\rangle\!\rangle\overline{M}$.
By Property 5.5, $\exists M' \in \overline{M}$, $\exists \sigma'$ such that $M_0[\sigma'\rangle M'$.
$[M_0\rangle$ strongly connected $\Longrightarrow$ $\exists \sigma''$ such that $M'[\sigma''\rangle M_0$. Thus, by applying Property 5.4, $\overline{M}[\![\overline{\sigma}''\rangle\!\rangle\overline{M}_0$. Hence, $[\![\overline{M}_0\rangle\!\rangle$ is strongly connected.

ii) $\Longrightarrow$ i) : Let $M_1 \in [M_0\rangle$. $\exists \sigma$ such that $M_0[\sigma\rangle M_1$.
From Property 5.4, $\exists \overline{\sigma}$ such that $\overline{M}_0[\![\overline{\sigma}\rangle\!\rangle\overline{M}_1$.
$[\![\overline{M}_0\rangle\!\rangle$ strongly connected $\Longrightarrow$ $\exists \overline{\sigma}'$ such that $\overline{M}_1[\![\overline{\sigma}'\rangle\!\rangle\overline{M}_0$.
From Property 5.5, $\forall M_1' \in \overline{M}_1$, $\exists \sigma''$, $M_1'[\sigma''\rangle M_0$. Hence we have a firing sequence leading from M_0 to M_0 and passing through M_1, and $[M_0\rangle$ is strongly connected.

The properties that we present in the next section are very closely related to the strong connection of the reachability graph. They concern the notion of *home states*, i.e., markings that can always be resumed by the net.

5.2.2 Home state

In a state graph, a home state is a state that can be reached from any other state by firing an appropriate sequence of transitions. This notion can be extended to the notion of home space. A home space is a set of states such that from each state of the graph, at least one state of the set can be reached.

Property 5.10 *The two following properties are equivalent.*

i) $\{M \in \overline{M}\}$ *is a home space for* $[M_0\rangle$,

ii) $\overline{M}$ *is a home state for* $[\![\overline{M}_0\rangle\!\rangle$.

Proof

i) $\Longrightarrow$ ii) : Let $\overline{M}_1 \in [\![\overline{M}_0\rangle\!\rangle$. $\exists\overline{\sigma}$ a firing sequence such that $\overline{M}_0[\![\overline{\sigma}\rangle\!\rangle\overline{M}_1$.
By Property 5.5, $\exists M_1' \in \overline{M}_1$, $\exists\sigma'$ such that $M_0[\sigma'\rangle M_1'$.
As we assume that $\{M \in \overline{M}\}$ is a home space, $\exists M'' \in \overline{M}$, $\exists\sigma''$ such that $M_1'[\sigma''\rangle M''$.
From Property 5.4, $\exists\sigma^{(3)}$ such that $\overline{M}_1[\![\sigma^{(3)}\rangle\!\rangle\overline{M}$. Hence, $\overline{M}$ is a home state for $[\![\overline{M}_0\rangle\!\rangle$.

ii) $\Longrightarrow$ i) : $\forall\overline{M}_1 \in [\![\overline{M}_0\rangle\!\rangle$, $\exists\overline{\sigma}$ such that $\overline{M}_1[\![\overline{\sigma}\rangle\!\rangle\overline{M}$.
From Property 5.5, $\forall M_1' \in \overline{M}_1$, $\exists M' \in \overline{M}$, $\exists\sigma'$ such that $M_1'[\sigma'\rangle M'$.
From Property 5.7, $M_1' \in \overline{M}_1$ and $\overline{M}_1 \in [\![\overline{M}_0\rangle\!\rangle \iff M_1' \in [M_0\rangle$.
Thus, we deduce $\forall M_1' \in [M_0\rangle$, $\exists M' \in \overline{M}$, $\exists\sigma'$ such that $M_1'[\sigma'\rangle M'$.
Hence $\{M \in \overline{M}\}$ is a home space for $[M_0\rangle$.

In the model of the philosophers, the SRG is strongly connected. Thus, every symbolic marking is a home state as it can be reached from every other symbolic marking. The RG is also strongly connected and thus every marking is a home state. Hence, the result in this particular case is stronger than Property 5.10. However, in the general case where the SRG is not strongly connected, if it contains a home state, we know that every marking of the RG can reach one of the markings represented by the home state, but they will not necessarily reach the same marking.

A home state is a state in which the system can always return. A state in which the system will necessarily return is called an *unavoidable home state*. In other words a net has an unavoidable home state if and only if there exists no infinite firing sequence that does not encounter that state.

Property 5.11 *The two following properties are equivalent.*

i) $\{M \in \overline{M}\}$ *is an unavoidable home space for* $[M_0\rangle$

ii) $\overline{M}$ *is an unavoidable home state for* $[\![\overline{M}_0\rangle\!\rangle$

Proof

i) $\Longrightarrow$ ii) : According to Property 5.10, $\overline{M}$ is a home state. We prove that it is unavoidable by contradiction. Let $\overline{M}_1 \in [\overline{M}_0\rangle$, and consider an infinite outgoing sequence from $\overline{M}_1$ that never encounters $\overline{M}$:

$$\overline{M}_1[\![t_{u_1},\overline{c}_{u_1}\rangle\!\rangle\overline{M}_2[\![t_{u_2},\overline{c}_{u_2}\rangle\!\rangle\overline{M}_3 \ldots\ \overline{M}_k[\![t_{u_k},\overline{c}_{u_k}\rangle\!\rangle \ldots$$

By Property 5.5, $\forall M_1' \in \overline{M}_1$, $\exists$ an infinite sequence such that :

$$M_1'[t_{u_1},c_{u_1}'\rangle M_2[t_{u_2},c_{u_2}'\rangle M_3 \ldots\ M_k[t_{u_k},c_{u_k}'\rangle \ldots$$

with $M_i \in \overline{M}_i$ and $\overline{M}_i \neq \overline{M}$.
However $\overline{M}_1 \in [\overline{M}_0\rangle$. By Property 5.7 $\forall M'_1 \in \overline{M}_1,\ M'_1 \in [M_0\rangle$. Hence, there exists an infinite outgoing sequence in $[M_0\rangle$ that never encounters $\{M \in \overline{M}\}$, which is in contradiction with the assumption that $\{M \in \overline{M}\}$ is an anavoidable home space for $[M_0\rangle$.

ii) $\Longrightarrow$ i) : According to Property 5.10, $\{M \in \overline{M}\}$ is a home space. We prove that it is unavoidable by contradiction. Let $M_1 \in [M_0\rangle$. Assume that there exists an infinite outgoing sequence from M_1 that never encounters $\{M \in \overline{M}\}$:

$$M_1[t_{u_1}, c_{u_1}\rangle M_2[t_{u_2}, c_{u_2}\rangle M_3 \ldots\ M_k[t_{u_k}, c_{u_k}\rangle \ldots$$

with $M_i \notin \overline{M}$.
According to Property 5.4 there exists an infinite sequence such that

$$\overline{M}_1[\![t_{u_1}, \overline{c}_{u_1}\rangle\!\rangle \overline{M}_2[\![t_{u_2}, \overline{c}_{u_2}\rangle\!\rangle \overline{M}_3 \ldots\ \overline{M}_k[\![t_{u_k}, \overline{c}_{u_k}\rangle\!\rangle \ldots$$

with $M_i \in \overline{M}_i$, hence $\overline{M}_i \neq \overline{M}$.
By Property 5.7, $M_1 \in [M_0\rangle \Longrightarrow \overline{M}_1 \in [\![\overline{M}_0\rangle\!\rangle$. Hence there must exist an infinite sequence in $[\![\overline{M}_0\rangle\!\rangle$ that never encounters $\overline{M}$ so that $\overline{M}$ is not unavoidable.

Marking $\overline{M}_1$ is an unavoidable home state of the SRG of the philosophers : every infinite sequence will encounter this marking. There is no unavoidable home state in the RG. However, there is no infinite sequence that does not encounter one of the markings $M_i, i = 1, \ldots, 5$. Thus, this set of markings, which is the set of markings represented by $\overline{M}_1$, is an unavoidable home space.

5.3 Liveness

There exist different notions related to liveness in state graphs. A state graph is pseudo-live (or deadlock-free) if there is at least one outgoing transition from every state of the graph. A transition is quasi-live if there is at least one edge of the graph labelled by that transition. A transition is live if for every state of the net there is a possible sequence of outgoing transitions from that state such that the considered transition appears in the sequence.

We now study these properties on the SRG.

Property 5.12 *The two following properties are equivalent.*

i) $[M_0\rangle$ *is pseudo-live,*

ii) $[\![\overline{M}_0\rangle\!\rangle$ *is pseudo-live.*

Proof

i) $\Longrightarrow$ ii) : Let $\overline{M}$ be any marking in $[\![\overline{M}_0\rangle\!\rangle$. Then $\exists\overline{\sigma}$ a firing sequence such that $\overline{M}_0[\![\overline{\sigma}\rangle\!\rangle\overline{M}$.
From Property 5.5, $\exists M' \in \overline{M},\ \exists\sigma',\ \ M_0[\sigma'\rangle M'$.
But $[M_0\rangle$ is pseudo-live. Then $\exists(t, c)$ such that $M'[t, c\rangle$.
However $M'[t, c\rangle \Longrightarrow \overline{M}[\![t, \overline{c}\rangle\!\rangle$ by Property 5.1. Hence there is an outgoing transition from $\overline{M}$, and $[\![\overline{M}_0\rangle\!\rangle$ is pseudo-live.

iii) $\Longrightarrow$ i) : $[\![\overline{M}_0\rangle\!\rangle$ is pseudo-live $\Longrightarrow \forall\overline{M} \in [\![\overline{M}_0\rangle\!\rangle,\ \ \exists(t, \overline{c})$ such that $\overline{M}[\![t, \overline{c}\rangle\!\rangle$
Thus, $\forall\overline{M} \in [\![\overline{M}_0\rangle\!\rangle,\ \ \forall M' \in \overline{M},\ \ \forall c'$ with $\alpha_{\overline{M}}(s_M.c') = \overline{c},\ \ M'[t, c'\rangle$ by Property 5.2.
Applying Property 5.7, $\forall M' \in [M_0\rangle,\ \ \exists c'$ such that $M'[t, c'\rangle$.
Hence, $[M_0\rangle$ is pseudo-live.

Property 5.13 *Quasi-liveness.*
Let $orb(c) = \{c' \in C \mid \exists s \in S,\ c' = s.c\}$. *The two following propositions hold true.*

i) (t, c) *is quasi-live in* $[M_0\rangle \implies \exists c' \in orb(c)$ *such that* (t, c') *is quasi-live in* $[\![\overline{M}_0\rangle\!\rangle$

ii) $(t, \overline{c})$ *is quasi-live in* $[\![\overline{M}_0\rangle\!\rangle \implies \forall c' \in orb(\overline{c}),\ (t, c')$ *is quasi-live in* $[M_0\rangle$.

Proof

i) (t, c) is quasi-live in $[M_0\rangle \Longrightarrow \exists M \in [M_0\rangle$ such that $M[t, c\rangle$. The associated symbolic marking $\overline{M}$ belongs to $[\![\overline{M}_0\rangle\!\rangle$ according to Property 5.7.
From Property 5.4, we know that $M[t, c\rangle \implies \overline{M}[t, \overline{c}\rangle$, with $\overline{c} \in orb(c)$.
Hence, $\exists (t, c')$ quasi-live in $[\![\overline{M}_0\rangle\!\rangle$, with $c' \in orb(c)$.

ii) $(t, \overline{c})$ is quasi-live in $[\![\overline{M}_0\rangle\!\rangle \implies \exists \overline{M} \in [\![\overline{M}_0\rangle\!\rangle,\ \overline{M}[\![t, \overline{c}\rangle\!\rangle$.
Using the definition of symbolic firing we know that $\exists M''$ such that $\overline{M}[t, \overline{c}\rangle M''$. Let $c' \in orb(\overline{c})$ and $s \in S$ be a symmetry such that $s.\overline{c} = c'$. Then $s.\overline{M}[t, c'\rangle s.M''$.
From Property 5.7, any marking of the class of $\overline{M}$, and in particular $s.\overline{M}$ belongs to $[M_0\rangle$.
Since $s.\overline{M}[t, c'\rangle$, then (t, c') is quasi-live in $[M_0\rangle$.

Let us consider the firing of $(Take, ph_3)$. This firing is quasi-live in the RG as it can occur from state M_5 for instance. Now, ph_2 belongs to $orb(ph_3)$ as it can be obtained by applying an admissible permutation on ph_3 and $(Take, ph_2)$ is quasi-live in the SRG as it can be fired from $\overline{M}_1$. Vice-versa, starting from the information that $(Take, ph_2)$ is quasi-live in the SRG, we can easily verify that transition $Take$ can be fired in the RG for any color obtained by applying an admissible permutation on ph_2, namely for any philosopher.

Property 5.14 *Liveness.*
$(t, \overline{c})$ *is quasi-live in* $[\![\overline{M}_0\rangle\!\rangle$ *and* $\overline{M}_0$ *is a home state* $\Longrightarrow \forall c' \in orb(\overline{c}),\ (t, c')$ *is live in* $[M_0\rangle$.

Proof $(t, \overline{c})$ is quasi-live in $[\![\overline{M}_0\rangle\!\rangle \implies \forall c' \in orb(\overline{c}),\ (t, c')$ is quasi-live in $[M_0\rangle$, i.e., $\exists M \in [M_0\rangle$ such that $M[t, c'\rangle$. By Property 5.10, $\overline{M}_0$ is a home state $\implies$ M_0 is a home state. Hence, $\forall M' \in [M_0\rangle$, $\exists \sigma_1$ such that $M'[\sigma_1\rangle M_0$. Also, $\exists \sigma_2$ such that $M_0[\sigma_2\rangle M$. Finally, $M'[\sigma_1\sigma_2\rangle M[t, c'\rangle$ and (t, c') is live in $[M_0\rangle$.

5.4 Numerical Properties

The properties that we present in this section are useful only in case one is interested in numerical results from the SRG. For instance, as the RG is isomorphic to a Markov chain for stochastic Petri nets [15, 16], it can be used for performance evaluation purposes. Under some timing constraints, we have shown [17, 18] that the SRG is isomorphic to a lumped Markov chain, i.e., a Markov chain whose nodes are classes of states. As the coefficients of the lumped Markov chain depend on the number of outgoing arcs from a marking, we need to know how many ordinary arcs are represented by a symbolic firing. Moreover, the most important performance criteria depend on the steady-state probabilities of the markings. From the SRG, it is easy to know the probability of a representative, but we need the number of markings it represents to derive the probability of each marking (markings are equally likely within an equivalence class).

Property 5.15 *Number of arcs : Let* $\overline{M}_1$ *and* $\overline{M}_2$ *be two symbolic markings. Let* $M_1' \in \overline{M}_1$. *We denote a set of ordinary arcs that lead from* M_1' *to any marking in* $\overline{M}_2$ *by* $A_{M_1'\overline{M}_2}$. *We denote the set of symbolic arcs that lead from* $\overline{M}_1$ *to* $\overline{M}_2$ *by* $A_{\overline{M}_1\overline{M}_2}$.
Let υ *be the application from* $A_{M_1'\overline{M}_2}$ *to* $A_{\overline{M}_1\overline{M}_2}$ *which associates with an arc* $a = M_1'[t, c\rangle M_2'$ *the*

symbolic arc $\upsilon(a) = \overline{M}_1[\![t, \overline{c}\rangle\!\rangle\overline{M}_2$, $\overline{c}$ *being the representative of* c *in* $\overline{M}_1$.
The application υ *is such that the cardinality of the reciprocal image of a symbolic arc* $\overline{a} = \overline{M}_1[\![t, \overline{c}\rangle\!\rangle\overline{M}_2$, *denoted by* $|\upsilon^{-1}(\overline{a})|$ *verifies :*

$$|\upsilon^{-1}(\overline{a})| = \frac{|S_{\overline{M}_1}|}{|\{s \in S_{\overline{M}_1} \mid s.\overline{c} = \overline{c}\}|}$$

Proof $\overline{M}_1[\![t, \overline{c}\rangle\!\rangle\overline{M}_2 \iff \exists M_2'$ such that $\overline{M}_1[t, \overline{c}\rangle M_2'$ according to the definition of symbolic firing $\iff \forall s \in S_{\overline{M}_1}$, $\overline{M}_1[t, s.\overline{c}\rangle s.M_2'$. Thus, the arcs represented by the symbolic firing we are considering are equal within a permutation in $S_{\overline{M}_1}$. However two permutations may lead to the same arc. What we are interested in is the set of different colours that can be reached from $\overline{c}$ using a permutation $s \in S_{\overline{M}_1}$. Let $orbit(\overline{c})$ be such a set. Assume $S(c) = \{s \in S_{\overline{M}_1} | s.c = c\}$. We obtain

$$|S_{\overline{M}_1}| = \sum_{c' \in orbit(\overline{c})} |S(c')|$$

Since $c' \in orbit(\overline{c})$, then $\exists s_1 \in S_{\overline{M}_1}$ such that $c' = s_1.\overline{c}$.
$s \in S(\overline{c}) \implies (s_1 \circ s \circ s_1^{-1}) \in S(c')$ and $s \in S(c') \implies (s_1^{-1} \circ s \circ s_1) \in S(\overline{c})$
Hence $\forall c' \in orbit(\overline{c})$, $|S(c')| = |S(\overline{c})|$, and $|S_{\overline{M}_1}| = |orbit(\overline{c})|.|S(\overline{c})|$.

Let us consider in the model of the philosophers the possible firings from marking M_0 that lead to a marking represented by $\overline{M}_1$. There are five such firings, all of them represented by a single symbolic firing :

$$\forall i = 1, \ldots, 5, \quad \upsilon(M_0[Take, ph_{i \ mod \ 5}\rangle M_i) = \overline{a} = \overline{M}_0[[Take, ph_0\rangle\rangle\overline{M}_1$$

Hence, the cardinality of the reciprocal image of $\overline{a}$ is 5. Now, among the admissible symmetries belonging to $S_{\overline{M}_0}$, i.e., that leave $\overline{M}_0$ unchanged, only the identity leaves ph_0 unchanged, and the denominator of the fraction is equal to 1. As every admissible symmetry leaves $\overline{M}_0$ unchanged, $|S_{\overline{M}_0}| = 5$ and the property is true.

Property 5.16 *Cardinality of a marking : The number of markings represented by* $\overline{M}$ *is given by :*

$$|\overline{M}| = \frac{|S|}{|S_{\overline{M}}|}$$

Proof Let $M' \in \overline{M}$ and (M') be the set of symmetries that lead from $\overline{M}$ to M'.

$$(M') = \{s \in S \mid s.\overline{M} = M'\}$$

Let $M'' \in \overline{M}$. There exists (at least) a permutation s_1 such that $s_1.M' = M''$. Therefore $s \in (M') \implies (s_1 \circ s) \in (M'')$, and vice versa, $s \in (M'') \implies (s_1^{-1} \circ s) \in (M')$. Hence $\forall M', M'' \in \overline{M}$, $|(M')| = |(M'')| = |(\overline{M})| = |S_{\overline{M}}|$.

$$|S| = \sum_{M' \in \overline{M}} (M') = |\overline{M}|.|S_{\overline{M}}|$$

In our example, the admissible symmetries are the rotations on color classes, hence $|S| = 5$. All of them leave marking $\overline{M}_0$ unchanged, but only the identity leaves markings $\overline{M}_1$ and $\overline{M}_2$ unchanged. Hence, $|S_{\overline{M}_0}| = 5$, whereas $|S_{\overline{M}_1}| = |S_{\overline{M}_2}| = 1$. We can check on the graphs that $\overline{M}_0$ represents a single marking and both $\overline{M}_1$ and $\overline{M}_2$ represent 5 markings.

6 Conclusions

We have presented the symbolic reachability graph for coloured Petri nets as a means to exploit model symmetries to improve their behavioural analysis efficiency. The SRG is defined for unrestricted coloured nets, while its construction procedure becomes completely effective (and can thus be implemented in a general algorithmic form) by introducing some syntactic restrictions. In particular a general algorithm was proposed in [10] for Well-formed Nets. Here we proved that most interesting behavioural as well as quantitative properties can be studied on the SRG rather than the ordinary RG without any loss of information. Of course the actual benefit of studying properties on the SRG rather than on the ordinary RG is related to the degree of symmetry of the model that determines the relative sizes of the two graphs. In case of models without any symmetry the two graphs are identical. In some cases of highly symmetric systems instead the size of the SRG may be virtually independent of the cardinality of the colour sets. In some other cases, the size of the SRG may increase much slower than the size of the RG as a function of the cardinality of the colour sets. On the average case in which the modeller chooses the coloured net formalism, some symmetry is inherently present in the model and the SRG may contain a number of nodes that is a few orders of magnitude lower than the ordinary RG, thus yielding substantial practical advantages.

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