Bounds for rewards of systems with clients/servers interaction

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Abstract. This paper presents a new method to compute bounds of performance parameters of Markov chains exhibiting a partition of the state space with some family of subsets visited in a sequential order. We use this structure to compute bounds of steady state reward rates on these subsets without computing the global steady state probabilities of the whole chain. The method presented is based on a combination of an aggregation procedure on the subsets and a strong stochastic ordering on the resulting aggregated space.

1 Introduction

Performance models based on Continuous Time Markov Chains (CTMC) have proven their usefulness for many years. In this context, performance measures can be frequently expressed as (steady state) expected reward rates (or instantaneous reward in steady state) $\mathcal{R} = \sum_{s\in\mathcal{S}^G} r(s)\pi[s]$ where $r(s)$ is the reward rate associated with the state s and π the (row) vector of the steady state probabilities of the CTMC $\mathcal{M} = (\mathcal{S}^G, Q, \pi_0)$ with state space $\mathcal{S}^G$, generator Q and initial probabilities π_0. Unfortunately, it is often impossible to compute π with an analytical method due to the complexity of the interactions among the entities of the system (there is no "closed" form for π). In these situations, we are lead to numerically solve the linear system $\pi Q = 0$. There are, however, well known difficulties for solving this equation, among these the size of the state space, and for many systems, the stiffness of the linear system when there are rare events in the modelled system. To cope with these problems, state space reduction methods have been proven very powerful. To compute the exact solution, reduction methods usually involve the whole state space. These methods are based on the structure of the behaviour of the system, like tensor based methods, or on exact Markovian aggregation (lumping methods). Another large class of methods is based on approximate solutions. The price to pay for the efficiency of the computation is then the difficulty to appreciate the quality of the result. Finally, bounding methods, often based on partial reduction of the chain, are a tradeoff

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between the two previous ways, providing both efficient solution methods and quantifiable estimation of the quality of the results.

Whatever the bounding method, it is always guided by the structure of the model, deduced from the properties of the studied system. An important structural property of $\mathcal{M}$ is the possibility to factorize the state space into a partition of N subspaces $\mathcal{S}_k^G$, allowing the decomposition $\mathcal{R} = \sum_{k=1}^{N} \mathcal{R}_k$. Depending on the properties of $\mathcal{M}$ with respect to the $\mathcal{S}_k^G$, several methods have been devised, mainly based on the Courtois results [2, 3] and/or stochastic orderings [10].

In this paper, we are concerned with systems exhibiting *sequences* of elementary actions such that, in each sequence, there is a non null probability to reach the next action. Examples of such systems are those with sequences of client/server like interactions, repairable systems, data base systems. Many authors, for instance [8, 6, 5, 1], have studied such systems and it has been shown that beside bounding the total probability of states corresponding to a given sequence (i.e. $\pi(\mathcal{S}_k^G)$), one of the main problems is to obtain bounds of the probability distribution inside a $\mathcal{S}_k^G$. We present here a new method to bound the conditional steady state reward rates on the subsets of states corresponding to these sequences of actions. Towards this goal, we extend the strong stochastic ordering between Markovian processes to aggregated versions of the processes from structural information about the Markov chain, *without computing the bounded aggregated chain*.

The organization of the paper is as follows: in Section 2 we present the context of the work and our approach. In Section 3, we introduce a first method to compute lower bounds of rewards rates on subsets of states, together with results about strong stochastic ordering and aggregation. We also define a general application framework and we study a detailed example with numerical results. In Section 4, a second method is proposed which extends results to other classes of systems and we explain in details computations of the bounds through an example. Section 5 concludes the paper. An extended version of this paper, with detailed numerical results is available [4].

2 Approach

We are interested in systems with client and server entities and activities which may be classified as autonomous client's activities and interaction activities between clients and servers. Whatever the (high level) modelling formalism of such systems (QN, SPN, ...), this leads to CTMCs where it is possible to identify sets $\mathcal{S}_k^G$ corresponding to these interactions. In the present study, we indeed work at the Markov chain level. In many important cases, although the state space of interactions is large due to their complexity, the corresponding reward remains "small" with regard to the contribution of the autonomous activities. This is also the case for dependability models, where "client/server interactions" states are the partial or total failure states of the repairable system. It is then meaningful to only compute bounds of the reward of these states.

If the whole state space of the chain is $\mathcal{S}^G = \biguplus_{k=1}^{N} \mathcal{S}_k^G$, then the reward is $\mathcal{R} = \sum_{k=1}^{N} \mathcal{R}_k$ and $\mathcal{R}_k = \pi(\mathcal{S}_k^G) \sum_{s \in \mathcal{S}_k^G} r(s)\pi(s|\mathcal{S}_k^G)$. We can regard the quantity $\mathcal{R}_{/k} = \sum_{s \in \mathcal{S}_k^G} r(s)\pi(s|\mathcal{S}_k^G)$ as the (steady state) conditional reward rate of the measure on $\mathcal{S}_k^G$. To obtain bounds for $\mathcal{R}$ we may then compute bounds for $\mathcal{R}_{/k}$ and for $\pi(\mathcal{S}_k^G)$. Note however that for some important models [8], only $\mathcal{R}_{/k}$ are needed. In this paper we concentrate ourselves on the first problem (obviously, these bounds must be computed *without* computing the distribution π on the $\mathcal{S}_k^G$). More specifically, as showed by many authors [8, 6, 7, 1], the most difficult problem is frequently to derive a lower bound of $\mathcal{R}_{/k}$ so that we only study this problem in the following.

In the rest of the paper, we rename, for ease of reading, $\mathcal{R}_{/k}$ as R and $\mathcal{S}_k^G$ as S, so that we study the computation of a lower bound R^- of $R = \sum_{s \in S} r(s)\pi(s|S)$.

Note that, in this paper, we assume that there is a single entry point in S, corresponding to the beginning of service. In fact, if this is not the case, Courtois has shown that there is a family $(\beta_{s'})$ of positive reals with $\sum_{s' \in in(S)} \beta_{s'} = 1$ such that $\pi(s|S) = \sum_{s' \in in(S)} \beta_{s'} \pi^{s'}(s|S)$ where $in(S)$ is the set of entry points of S and $\pi^{s'}$ the steady state probabilities of the modified chain with all input transitions of S "redirected" to s'. Hence, we may derive a bound R^- from bounds $R^{s'-}$.

Evidently, a first lower bound of R is $min_{s \in S}\{r(s)\}\pi(S)$. This is obviously very poor, since if $\exists s : r(s) = 0$ then the lower bound is zero. To go further, we suppose that, once again, the space S itself, possesses a structure which will help to define tighter bounds. If interactions are sequences of elementary actions, S may be partitioned into subsets $(S_i)_{1 \le i \le n}$ such that r is constant in each S_i. If $r(s) = r_i$ for $s \in S_i$, then $R = \sum_{i=1}^{n} r_i \pi(S_i|S)$ (with $\pi(A|B) = \sum_{s \in A} \pi(s|B)$).

We propose two methods to find a lower bound of R, which are based on an extension of the strong stochastic ordering of Markov chains to aggregated versions of these chains (see Proposition 1) without computing the bounded aggregated chain. We call "adapted bounding matrices" (see Definition 1) matrices which allow such an extension. The first method bounds the conditional probabilities $\pi(.|S)$ themselves. As we shall see in Section 3 we obtain $R^- > 0$ under conditions discussed at the end of the section. To cope with systems which do not satisfy the required conditions of method 1, we propose in Section 4 a second method, which relaxes these conditions but needs more structural properties.

3 First method

Let us recall that we want to compute a lower bound of $R_S = \sum_{s \in S} r(s)\pi(s|S)$ where S is a subspace of the whole state space $\mathcal{S}$ of a Markov chain. Moreover we assume that S may be partitioned into n subsets S_i such that $S_i = \{s \in S \mid r(s) = r_i\}$, so that $R_S = \sum_{i=1}^{n} r_i \pi(S_i|S)$. In this section we shall directly lower bound each $\pi(S_i|S)$ using properties of substochastic matrices. From these bounds $\widehat{\pi}^-[i]$, we can derive the lower bound $R_S^- = \sum_{i=1}^n r_i \widehat{\pi}^-[i]$ with $R_S^- > 0$ under conditions explained at the end of the section.

We then present a general class of systems for which our method is applicable, followed by an effective example of such a system.

3.1 Idea of the method

Our main objective is to obtain bounds $\widehat{\pi}^-[i]$ using only the parameters of the model (for instance the generator Q of the chain) and not the steady state probabilities on S which will not be computed. To this end, we first restate the problem in the framework of DTMCs. As usual with the uniformization method, we take $P = I_{|S|} + \frac{1}{M} Q_{|S}$ where $M \geq max_{s \in \mathcal{S}}\{-Q[s,s]\}$ and $Q_{|S}$ is the restriction of Q to S.

Since, for DTMCs, we can establish a link between the steady state probabilities and the matrix P via the so called visit-ratios, we define bounds for $\pi(S_i|S)$ from bounds of matrices derived from the matrix P.

Denoting by s_0 the unique entry state in S from the rest of the whole state space, it is well known that

$$\pi(s|S) = \frac{V(s_0,s)}{\sum_{s' \in S} V(s_0,s')} \tag{1}$$

where $V(s_0,s)$ denotes the mean number of visits to s from s_0 before leaving S. To get a lower bound of $\pi(S_i|S)$, it is then sufficient to compute a lower bound of $\sum_{s \in S_i} V(s_0,s)$ and an upper bound of $\sum_{s' \in S} V(s_0,s')$. The link with the matrix P lies in the relation $V(s_0,s) = \left(\sum_{k \geq 0} P^k\right)[s_0,s]$

A majorization of $\sum_{s'} V(s_0,s')$ by a direct (componentwise) majorization of P may, in the general case, provide a non strictly substochastic $|S| \times |S|$ matrix P^+ so that the power sum of P^+ would have no meaning. Hence, we compute a majorization of $\pi(S_i|S)$ by an upper bounding aggregation on the (S_i), with a $n \times n$ matrix $\widehat{P}^+$, the bounding being "compatible" with the power sum operator. On the other hand, we compute a minorization of $\pi(S_i|S)$ by a lower bounding (again a $n \times n$ matrix) of $P[s_i, S_j] = \sum_{s_j \in S_j} P[s_i, s_j]$ $\forall s_i \in S_i$. Another advantage of the introduction of these two $n \times n$ matrices is that, since we compute powers of matrices, smaller matrices will allow faster and more accurate computations.

Note that, in fact, we may choose different partitions (S_i) for the upper bounding of $\sum_{s'} V(s_0,s')$ and the lower bounding of $V(s_0,s)$. However, for ease of reading, we keep the same partition for the rest of this section.

3.2 Bounding substochastic matrices

In this section, we provide a mean to define bounds for a substochastic matrix and its power series. The following lemma sets out sufficient conditions for a $n \times n$ matrix to lower bound $V(s_0, S_i) = \sum_{s_i \in S_i} V(s_0, s_i)$.

Lemma 1. *Let $\widehat{P}^-$ be a $n \times n$ matrix such that $\forall i,j,\ \forall s_i \in S_i,\ \widehat{P}^-[i,j] \leq P[s_i, S_j]$. Then*

$$\left(\sum_{k\geq 0} (\widehat{P}^-)^k\right)[i,j] \leq min_{s_i\in S_i}\{V(s_i,S_j)\} \tag{2}$$

Proof. Since $V(s_i,S_j) = \left(\sum_{k\geq 0} P^k\right)[s_i,S_j]$, let us prove by induction that for any k: $(\widehat{P}^-)^k[i,j] \leq min_{s_i\in S_i}\{P^k[s_i,S_j]\}$.

The property is clearly true from the hypothesis for $k = 1$. Assuming that the property is true for k, we have

$$\begin{aligned} P^{k+1}[s_i,S_j] \stackrel{def}{=} \sum_{s_j\in S_j}\sum_{l=1}^{n}\sum_{s_l\in S_l} P^k[s_i,s_l].P[s_l,s_j] &= \sum_{l=1}^{n}\sum_{s_l\in S_l} P^k[s_i,s_l].P[s_l,S_j] \\ &\geq \sum_{l=1}^{n} (\widehat{P}^-)^k[i,l].\widehat{P}^-[l,j] \end{aligned}$$

from the hypothesis and the property for k.

In contrast with the minorization, the majorization of $V(s_0,S_i)$ imposes some constraints on the bounding matrix. Adaptation is an extension of the strong stochastic ordering [10] concept between Markovian processes to $\widehat{P}^+$ and an aggregation of P on the (S_i). Monotonicity, like for Markovian processes, ensures the expansion of an initial strong stochastic ordering to any (discrete) time, by transitions of the chain.

Definition 1. *A $n\times n$ matrix X is* monotonic *iff* $\forall 1 \leq i < j \leq n,\ \forall 1 \leq m \leq n,\ \ \sum_{l=1}^{m} X[i,l] \geq \sum_{l=1}^{m} X[j,l]$

A $n \times n$ matrix $\widehat{P}^+$ is an adapted (upper) bound *of P with respect to the partition $(S_i)_{1\leq i\leq n}$ iff (i) it is monotonic and (ii)* $\forall 1 \leq i \leq n,\ \forall s \in S_i,\ \forall 1 \leq m \leq n,\ \ \sum_{j=1}^{m}\sum_{s'\in S_j} P[s,s'] \leq \sum_{j=1}^{m} \widehat{P}^+[i,j]$

Adaptation is a sufficient condition to upper bound the visit-ratios, as stated in the next proposition which is the equivalent to the result for Markovian processes.

Proposition 1. *If $\widehat{P}^+$ is an adapted upper bound of P, then for any integer k*

$$\forall 1 \leq i \leq n,\ \forall s \in S_i,\ \forall 1 \leq m \leq n,\quad \sum_{j=1}^{m}\sum_{s'\in S_j} P^k[s,s'] \leq \sum_{j=1}^{m} (\widehat{P}^+)^k[i,j] \tag{3}$$

Proof. The proof is done by induction.

• For $k = 0$, if $i > m$ the inequality resumes to $0 \leq 0$, and if $i \leq m$, it resumes to $1 \leq 1$.

• Let us assume that the property is true for k.

$$A = \sum_{j=1}^{m} (\widehat{P}^{+})^{k+1}[i,j] = \sum_{j=1}^{m}\sum_{l=1}^{n} (\widehat{P}^{+})^{k}[i,l]\widehat{P}^{+}[l,j] = \sum_{l=1}^{n} (\widehat{P}^{+})^{k}[i,l] \sum_{j=1}^{m} \widehat{P}^{+}[l,j]$$

Using a classical transformation

$$\begin{aligned} A = \quad & \left[\sum_{l=1}^{n} (\widehat{P}^{+})^{k}[i,l]\right] \left[\sum_{j=1}^{m} \widehat{P}^{+}[n,j]\right] \\ & \vdots \\ & + \left[\sum_{l=1}^{t} (\widehat{P}^{+})^{k}[i,l]\right] \left[\sum_{j=1}^{m} \widehat{P}^{+}[t,j] - \sum_{j=1}^{m} \widehat{P}^{+}[t+1,j]\right] \\ & \vdots \\ & + \left[\sum_{l=1}^{1} (\widehat{P}^{+})^{k}[i,l]\right] \left[\sum_{j=1}^{m} \widehat{P}^{+}[1,j] - \sum_{j=1}^{m} \widehat{P}^{+}[2,j]\right] \end{aligned}$$

By monotonicity of $\widehat{P}^{+}$, each second term of the products is positive, hence we may apply the inductive inequality on the first terms for $s \in S_i$

$$\begin{aligned} A \geq B = \quad & \left[\sum_{l=1}^{n} \sum_{s' \in S_l} P^{k}[s,s']\right] \left[\sum_{j=1}^{m} \widehat{P}^{+}[n,j]\right] \\ & \vdots \\ & + \left[\sum_{l=1}^{t} \sum_{s' \in S_l} P^{k}[s,s']\right] \left[\sum_{j=1}^{m} \widehat{P}^{+}[t,j] - \sum_{j=1}^{m} \widehat{P}^{+}[t+1,j]\right] \\ & \vdots \\ & + \left[\sum_{l=1}^{1} \sum_{s' \in S_l} P^{k}[s,s']\right] \left[\sum_{j=1}^{m} \widehat{P}^{+}[1,j] - \sum_{j=1}^{m} \widehat{P}^{+}[2,j]\right] \end{aligned}$$

We now apply the inverse transformation to B: $B = \sum_{l=1}^{n} \left[\sum_{s' \in S_l} P^{k}[s,s']\right] \left[\sum_{j=1}^{m} \widehat{P}^{+}[l,j]\right]$ and since $\widehat{P}^{+}$ is an adapted upper bound of P, $B \geq C = \sum_{l=1}^{n} \left[\sum_{s' \in S_l} P^{k}[s,s']\right] \left[\sum_{j=1}^{m} \sum_{s'' \in S_j} P[s',s'']\right]$. Rearranging the order of summation, we get $C = \sum_{j=1}^{m} \sum_{s'' \in S_j} \sum_{l=1}^{n} \sum_{s' \in S_l} P^{k}[s,s']P[s',s''] = \sum_{j=1}^{m} \sum_{s'' \in S_j} P^{k+1}[s,s'']$

3.3 Bounds for conditional steady state probabilities

Gathering previous results, we can state the following theorem which provides a lower bound of $\pi(S_i|S)$. Then we explain how to obtain an upper bound $\pi(S_i|S)$ using these lower bounds and relations established above.

Theorem 1. *Let $\widehat{P}^-$ and $\widehat{P}^+$ be the matrices of lemma 1 and proposition 1, and let $\widehat{P}^+$ is strictly substochastic. Then, for any i*

$$\pi(S_i|S) \geq \frac{\left(\sum_{k\geq 0}(\widehat{P}^-)^k\right)[1,i]}{\sum_{j=1}^{n}\left(\sum_{k\geq 0}(\widehat{P}^+)^k\right)[1,j]} \tag{4}$$

Proof. From (1), and choosing $i = 1$ and $s_i = s_0$ we apply lemma 1 and proposition 1 with $m = n$ and taking the limit in the sum over k.

From the previous bounds, we can now compute an upper bound of $\pi(S_i|S)$. The derivation of this bound is similar to the complement property used for probability distributions: $\Pr(S\backslash A) = 1 - \Pr(A) \Rightarrow \Pr(A) \leq 1 - \Pr^-(S\backslash A)$ with $\Pr^-(S\backslash A) \leq \Pr(A)$.

Let $V = \sum_{s\in S} V(s_0, s)$, we have from proposition 1, (with $\widehat{P}^+$ strictly substochastic): $\sum_{j=1}^{i} V(s_0, S_j) \leq \sum_{j=1}^{i}\left(\sum_{k\geq 0}(\widehat{P}^+)^k\right)[1,j] \stackrel{def}{=} V_i^+$. From lemma 1, $V \geq \sum_{j=1}^{n}\left(\sum_{k\geq 0}(\widehat{P}^-)^k\right)[1,j] \stackrel{def}{=} V^-$. Since $\pi(S_i|S) = \frac{\sum_{s\in S_i} V(s_0,s)}{V}$, $\sum_{j=1}^{i}\pi(S_j|S)$ $= \sum_{j=1}^{i}\frac{V(s_0,S_j)}{V} \leq \frac{V_i^+}{V^-}$. Hence

$$\pi(S_i|S) \leq \max\{\frac{V_i^+}{V^-} - \sum_{j=1}^{i-1}\pi^-(S_j|S), 1 - \sum_{j\neq i}\pi^-(S_j|S)\} \tag{5}$$

where $\pi^-(S_j|S)$ is the lower bound given by the theorem 1.

The effective computation of the above bounds of $\pi(S_i|S)$ involves the numerical calculation of two power series of matrices which are performed with standard methods [9].

3.4 A general application framework

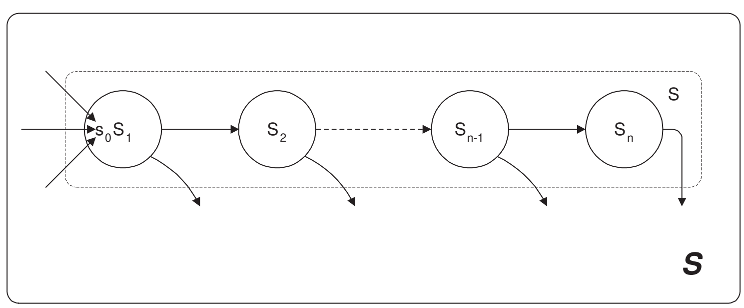

Fig. 1. Structure of the state space of the general application framework (method 1)

In this section we show how to build matrices $\widehat{P}^-$ and $\widehat{P}^+$ for a large class of systems. The structure of these systems must be such that we should be able to derive the $\widehat{P}^-$ and $\widehat{P}^+$ matrices directly from the model. Moreover, the memory requirements for the computation of $\widehat{P}^-$ and $\widehat{P}^+$ must be polynomial, or even linear, with respect to the size of the model. We assume that the structure of the state space S is as depicted in Figure 1 and that there is only one entry state in S, which belongs to S_1. Finally, we impose that the system may leave S from each state of S_n.

The definition of $\widehat{P}^-$ is the easiest one. We simply take

$$\widehat{P}^- = \begin{pmatrix} * & * & & & \\ & * & * & & \\ & & \ddots & \ddots & \\ & & & \ddots & * \\ & & & & * \end{pmatrix} \quad \text{with } \widehat{P}^-[i,j] \begin{cases} \leq \min_{s\in S_i} \sum_{s'\in S_j} P[s,s'] & \text{if } j = i \\ & \text{or } j = i+1 \\ = 0 & \text{otherwise} \end{cases}$$

It is clear that $\widehat{P}^-$ fulfills the hypothesis of lemma 1.

The idea behind the definition of an upper bound $\widehat{P}^+$ is to lower the transition probabilities of leaving S and to lower the "advance" from S_i to S_{i+1}. Let us suppose that the structure of the system allows us to derive to sequences (L_i) and (N_i) with the following properties. Let us denote by $\mathit{Leave}(s) = 1 - \sum_{s'\in S\setminus\{s\}} P[s,s']$ the transition probability of leaving S from s, then $\forall 1 \leq i \leq n,\ 0 \leq L_i \leq \min_{s\in S_l, l\geq i}\{\mathit{Leave}(s)\}$ ($L_n > 0$ since the chain may leave S with a non null probability from each state of S_n). L_{n-1} lower bounds the transition probability for the chain to leave S from S_{n-1} or S_n and L_1 lower bounds the transition probability for the chain to leave S. (L_i) is obviously an increasing sequence.

The N_i (Next), instead, has to lower bound the transition probabilities from any state of S_i to the set S_{i+1}: $0 < N_i \leq \min_{s\in S_i} \sum_{s'\in S_{i+1}} P[s,s']$. Note that we propose the condition $0 < N_i$ rather than $0 \leq N_i$, which seems quite natural, and provides a sufficient condition for $\widehat{P}^+$ to be strictly substochastic (see the proof of lemma 2). With these conditions, we define the matrix

$$\widehat{P}^+ = \begin{pmatrix} 1-L_1-N_1 & N_1 & & & \\ & 1-L_2-N_2 & N_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & N_{n-1} \\ & & & & 1-L_n \end{pmatrix} \quad \begin{array}{c} \text{with } \widehat{P}^+[i,j] = 0 \text{ if} \\ j \neq i \text{ or } j \neq i+1 \end{array}$$

Lemma 2. *$\widehat{P}^+$ is strictly substochastic and is an adapted upper bound of P with respect to the partition $(S_i)_{1\leq i\leq n}$.*

Proof. • $\widehat{P}^+$ is strictly substochastic: for $1 \leq i \leq n-1$: $\widehat{P}^+[i,i] = 1-L_i-N_i \geq 1-\min_{s\in S_l,l\geq i}\{1-\sum_{s'\in S} P[s,s']\}-\min_{s\in S_i}\sum_{s'\in S_{i+1}} P[s,s'] = \max_{s\in S_l,l\geq i}\sum_{s'\in S} P[s,s']-$

$\min_{s\in S_i}\sum_{s'\in S_{i+1}} P[s,s'] \geq \max_{s\in S_i}\sum_{s'\in S} P[s,s'] - \min_{s\in S_i}\sum_{s'\in S_{i+1}} P[s,s'] \geq 0$

For all other values of j, clearly $\widehat{P}^{+}[i,j] \geq 0$. Moreover $\sum_j \widehat{P}^{+}[i,j] = 1 - L_i \leq 1$. Since $1 - L_n < 1$, and $\forall i \leq n-1,\ N_i > 0$, $\widehat{P}^{+}$ is strictly substochastic.

• $\widehat{P}^{+}$ is monotonic: we have to show that for $i < j$ and m given: $A = \sum_{l=1}^{m} \widehat{P}^{+}[i,l] \geq \sum_{l=1}^{m} \widehat{P}^{+}[j,l] = B$ and by transitivity, it is sufficient to prove the property for $j = i+1$. We have the following situations:

m	A	B
$m < i$	0	0
$m = i$	$\widehat{P}^{+}[i,i]$	0
$m = i+1$	$1 - L_i$	$1 - L_{i+1} - N_{i+1} \leq 1 - L_{i+1}$ and $A \geq B$ since (L_i) is increasing
$m > i+1$	$1 - L_i$	$1 - L_{i+1}$ (same as above)

• $\widehat{P}^{+}$ is an adapted upper bound of P: given i, j, m and $s \in S_i$, let us prove that $A = \sum_{j=1}^{m}\sum_{s'\in S_j} P[s,s'] \leq \sum_{j=1}^{m} \widehat{P}^{+}[i,j] = B$.

For $m < i$: $A = B = 0$.

For $m = i$: $A = \sum_{s'\in S_i} P[s,s'] = 1 - \sum_{s'\notin S} P[s,s'] - \sum_{s'\in S_{i+1}} P[s,s']$ from the properties of the chain in S, so that $A \leq 1 - (1 - \sum_{s'\in S} P[s,s']) - N_i \leq 1 - L_i - N_i = B$

For $m > i$: $A = \sum_{s'\in S_i} P[s,s'] + \sum_{s'\in S_{i+1}} P[s,s'] = 1 - \mathit{Leave}(s) \leq 1 - L_i = B$ again from the properties of the chain in S.

Above results were established with the hypothesis that we have only one entry point belonging to S_1. Courtois and Semal [2] showed that if we have a set $S_1^{(0)}$ of entry points in S_1, there are positive reals $(a_s)_{s\in S_1^{(0)}}$ such that the steady state distribution is a linear combination of the steady state distributions $\pi^{(s)}$ corresponding to the modified Markov chain where all inputs in S_1 are "redirected" to $s \in S_1^{(0)}$: $\pi = \sum_{s\in S_1^{(0)}} a_s \pi^{(s)}$, with $\sum_{s\in S_1^{(0)}} a_s = 1$. Since our bounds do not depend on the actual entry point in S_1, we see that they are still valid for all the distributions $\pi^{(s)}$, therefore for any distribution π corresponding to a chain without a unique entry point in S_1.

3.5 An example

We illustrate the general framework exposed above and we present numerical results for a detailed example of a system. In the framework of clients/servers systems, let us assume that clients request a service that is composed of two phases: computation of the requested results and transmission of these results from the server to the client. The computation may be done by nc distinct units with exponential service distribution with different rates α_l, and the transmission may be carried out in nt different ways, also with exponential service distribution with rates β_m. Therefore, probability distributions of computation and transmission phases are hyperexponential. Here, S is the set of states with

$n-1$ requests being served (either doing actual service or being transmitted) by a set of servers, S_i is the set of states where the results of $i-1$ client's requests of the $n-1$ are being transmitted, computed but not yet received by the client, and $n-i$ requests for which the server is still in the computation phase. From a state in S_i, a server may begin its transmission of results, which means that the chain leaves S_i and enters S_{i+1}, or a new request enters the service subsystem or one client has received its results, which means that the chain leaves S, or an action independent of the services occurs, which is translated by a transition of the chain inside S_i. We suppose that the system may actually leave the service area (i.e. S_n) when all requests are in the reception of results phase. It could not be the case: for example an Erlang distribution (with more than one stage) for transmission, generates (Markovian) entry states in S_n that do not fulfill this condition.

Computation of the bounding matrices First, let us note that expressions (4) and (5) for bounds do not depend on the actual value of M involved in the uniformization procedure. Since it is easy to compute one such value from the model of the system, we simply assume here that M is fixed.

We now show how to compute the matrices $\widehat{P}^-$ and $\widehat{P}^+$. Le us denote by $\alpha_{\min}, \alpha_{\max}, \beta_{\min}$ and $\beta_{\max}$ the respective minimum and maximum of the various rates of computation and transmission. We choose $\widehat{P}^-[i, i+1] = \frac{(n-i)\alpha_{\min}}{M}$ and since $\widehat{P}^-[i,i]$ must fulfill $\widehat{P}^-[i,i] \leq \min_{s \in S_i}\{1 - \frac{\sum_{s' \notin S_i} q_{s,s'}}{M}\}$, we set $\widehat{P}^-[i,i] = 1 - \frac{(n-i)\alpha_{\max} + (i-1)\beta_{\max} + x_{\max}}{M}$ where $x_{\max}$ denotes an upper bound of the arrival rate of new requests. A possible value for M is then $M = (n-1)\max\{\alpha_{\max}, \beta_{\max}\} + x_{\max} + 1$.

For what concerns $\widehat{P}^+$, we define the quantities L_i and N_i. Since contributions to *Leave*(s) come from internal (end of service of a client'request) and external (new request) events of S, we can minorize *Leave*(s) using a minorization of the arrival rates ($x_{\min}$). So we take $L_i = \frac{(i-1)\beta_{\min} + x_{\min}}{M}$. On the other side, we choose $N_i = \frac{(n-i)\alpha_{\min}}{M}$.

Complexity reduction Let us evaluate the reduction of the state space using the aggregated states corresponding to the S_i instead of the subsets S_i themselves. As we can choose one computation among nc and one transmission among nt, $|S_i| = \binom{n-i+nc-1}{n-1} \times \binom{i-1+nt-1}{i-1}$. Since $\min\{n-i, i-1\} \geq \lfloor \frac{n-1}{2} \rfloor$, we have, with increasing values of n, $\binom{n-i+nc-1}{n-1} = \Theta((n-1)^{nc-1}) = \Theta(n^{nc-1})$ or $\binom{i-1+nt-1}{i-1} = \Theta((i-1)^{nt-1}) = \Theta(n^{nt-1})$, that is to say, $|S_i| \geq \Theta(n^{\min\{nt,nc\}-1})$. For example, with $n = 100$, $nc = 5$ and $nt = 10$ we substitute one element for a subset S_i with about 10^5 elements.

Numerical results We have computed bounding vectors $\widehat{\pi}^-$ and $\widehat{\pi}^+$ for a simplified version of our example: we assume only two stages in the hyperexponential distribution for computation, and also two stages in the one for transmission. We have fixed $n = 10$ and $x_{\min} = x_{\max} = 0$. The first results are about the quality of the bounds, with respect to the values of $\alpha_{\min}, \alpha_{\max}, \beta_{\min}$ and $\beta_{\max}$. As quality criterion, we compare the value $c = \sum_{i=1}^{n} \widehat{\pi}^{-}[i]$ to 1: the more c is near from 1, the better will be the lower bound. To obtain an indication on the variation of c with respect to the rates α and β, we have computed c for different ratios $\Delta_\alpha = \frac{\alpha_{\max}-\alpha_{\min}}{\alpha_{\min}}$ and $\Delta_\beta = \frac{\beta_{\max}-\beta_{\min}}{\beta_{\min}}$ (0.1, 0.5, 1) as well as for various ratios $\frac{\alpha_{\min}}{\beta_{\min}}$ (0.1, 1, 10). Results are reported in table 1. It clearly appears, as

case	$\frac{\alpha_{\min}}{\beta_{\min}}$	Δ_α	Δ_β	$\sum_{i=1}^{n} \widehat{\pi}^{-}[i]$
1	0.01	0.1	0.1	0.9023
2	0.1	0.1	0.1	0.8660
3	0.1	0.1	0.5	0.7995
4	0.1	0.5	0.1	0.5970
5	0.1	0.5	0.5	0.5585
6	0.1	1	0.1	0.4224
7	1	0.1	0.1	0.7596
8	1	0.5	0.1	0.3846
9	10	0.1	0.1	0.5660
10	10	0.5	0.5	0.1453

Table 1. Summary of numerical results

expected, that the lower is $\frac{\alpha_{\min}}{\beta_{\min}}$, the better is the bound.

i	Δ^{-}	$\widehat{\pi}^{-}$	$\min_{s_0} \widehat{\pi}^{(s_0)}$	$\max_{s_0} \widehat{\pi}^{(s_0)}$	$\widehat{\pi}^{+}$	Δ^{+}
1	0.0766025	0.5407474	0.5856063	0.6039496	0.6747501	0.1172291
2	0.1640449	0.2457943	0.2940281	0.3005905	0.3797969	0.2635029
3	0.2127773	0.0662072	0.0841023	0.0922648	0.2002099	1.1699492
4	0.2555454	0.0117033	0.0157206	0.0186712	0.1457060	6.8037702
5	0.2942371	0.0014186	0.0020100	0.0025982	0.1354212	51.121
6	0.3298200	0.0001194	0.0001782	0.0002516	0.1341221	532.16846
7	0.3628701	0.0000069	0.0000108	0.0000167	0.1340096	8012.1245
8	0.3937673	0.	4.306E-07	7.304E-07	0.1340029	183468.89
9	0.4227813	0.	0.	0.	0.1340027	7083390.2
10	0.4501153	0.	0.	0.	0.1340027	6.073E+08

Table 2. Comparison between bounds and exact values for $\widehat{\pi}$ (case 2)

Given these results, we have computed the possible exact values $\widehat{\pi}^{(s_0)}$ corresponding to modified chain with entry point in $s_0 \in S_1$. We obtain $\widehat{\pi}^{(s_0)}$ by first computing the matrix P and then visit-ratios matrix V ($V[s, s'] = \left(\sum_{k\geq 0} P^k\right)[s, s']$) (all matrices are $|S| \times |S|$; in our example, $|S| = 220$). We have compared the vectors $\widehat{\pi}^{(s_0)}$ and $\widehat{\pi}^{-}$ for the first cases giving high c values: re-

i	Δ^-	$\widehat{\pi}^-$	$\min_{s_0} \widehat{\pi}^{(s_0)}$	$\max_{s_0} \widehat{\pi}^{(s_0)}$	$\widehat{\pi}^+$	Δ^+
1	0.1201605	0.5407474	0.6145978	0.6326504	0.7456626	0.1786329
2	0.2747293	0.2044843	0.2819420	0.2896483	0.4093995	0.4134366
3	0.3978016	0.0433919	0.0720558	0.0795974	0.2483071	2.1195382
4	0.5055382	0.0058865	0.0119049	0.0142554	0.2108017	13.787473
5	0.5968059	0.0005392	0.0013374	0.0017443	0.2054544	116.78769
6	0.6727950	0.0000340	0.0001038	0.0001479	0.2049492	1384.7932
7	0.7353948	0.0000015	0.0000055	0.0000086	0.2049167	23863.654
8	0.7865970	0.	0.	0.	0.2049153	626920.53
9	0.8282621	0.	0.	0.	0.2049152	27810027.
10	0.8620358	0.	0.	0.	0.2049152	2.743E+09

Table 3. Comparison between bounds and exact values for $\widehat{\pi}$ (case 3)

sults are presented in Tables 2 and 3. In these tables, $\Delta^-(i) = \frac{\widehat{\pi}^-[i] - \min_{s_0} \widehat{\pi}^{(s_0)}[i]}{\min_{s_0} \widehat{\pi}^{(s_0)}[i]}$ gives the relative error between $\min_{s_0 \in S_1} \widehat{\pi}^{(s_0)}[i]$ and $\widehat{\pi}^-[i]$. Likewise, $\Delta^+(i) = \frac{\widehat{\pi}^+[i] - \max_{s_0} \widehat{\pi}^{(s_0)}[i]}{\max_{s_0} \widehat{\pi}^{(s_0)}[i]}$. We can observe that the relative error for the lower bound increases with i, and less quickly in the case 2 than in the case 3.

Finally, let us note that, since we should obtain a non null lower bound, it is necessary that, either $r_1 \neq 0$ or $N_1 \neq 0$ if $r_1 = 0$, and more generally, $N_i \neq 0$. The condition $N_1 \neq 0$, also pointed out by other authors [7], means that, if the reward rate is null in S_1, the system must be able (i.e. with a non null probability) to enter S_2 (beginning of the last phase of service for one request) *from each state of* S_1. Like for S_n (and L_n) this may not be always the case. This justifies the introduction of another method which relaxes this constraint. We make up for the weakening of the conditions on S_1 with more structural information about the state space of the chain.

4 A more elaborated method

We derive in this section another lower bound of $R_S = \sum_{s \in S} r(s)\pi(s|S)$ where S is a subspace of the whole state space $\mathcal{S}$, partitioned in n subspaces S_i, so that $R_S = \sum_{i=1}^{n} r_i \pi(S_i|S)$.

In many situations, we are interested in systems for which the sequence (r_i) is *increasing*: the more the request is in progress, the greater is the associated reward. Hence, in contrast with the first method, we do not compute a componentwise minorization of $\widehat{\pi}_{/S}$ ($\widehat{\pi}_{/S}[i] = \pi(S_i|S)$). Instead, we define a probability vector $\widehat{\pi}^-$ such that $\widehat{\pi}^- <_{st} \widehat{\pi}_{/S}$ where $<_{st}$ denotes the strong stochastic ordering. Thus we get from $<_{st}$ properties

$$R_S^- \overset{def}{=} \sum_{i=1}^{n} r_i \widehat{\pi}^-[i] \leq \sum_{i=1}^{n} r_i \widehat{\pi}_{/S}[i] = R_S$$

The distribution $\widehat{\pi}^-$ is computed thanks to a refinement of the partition of S: each S_i is now itself partitioned into $(S_{i,j})_{1\le j\le n_i}$. Using the forward equations of the chain on S, and from results of the previous section, we define inductively $\widehat{\pi}^-[i]$ starting with $i=1$.

Here also, we could introduce different partitions $(S_{i,j})$ of a given S_i for upper and lower bounding of probabilities. We keep only one partition for both bounds for readability.

4.1 Discrete strong stochastic ordering

We restate in our specific context, the equivalence of two of the definitions of strong stochastic ordering between two probability distributions over $\{1,\ldots,n\}$ (proofs of the lemmas are omitted due to lack of space and may be found in [4]).

Lemma 3. *Let p and q be two probability distributions over $\{1,\ldots,n\}$ and $m\le n$, such that $\forall 1\le i\le m,\quad \sum_{j=1}^{i} p_j \ge \sum_{j=1}^{i} q_j \quad$ and $\quad \sum_{j=1}^{m} p_j = \sum_{j=1}^{m} q_j$. Then, for any increasing sequence $(r_i)_{1\le i\le m}$ of positive numbers: $\sum_{i=1}^{m} p_i r_i \le \sum_{i=1}^{m} q_i r_i$*

A sufficient condition to ensure the hypothesis of the previous lemma is given by the classical comparison criterion for sums of positive numbers.

Lemma 4. *Let p and q be two probability distributions over $\{1,\ldots,n\}$ such that $\sum_{j=1}^{m} p_j = \sum_{j=1}^{m} q_j$ $(m\le n)$. If, $\forall 1<j\le m$, $\frac{p_j}{p_{j-1}} \le \frac{q_j}{q_{j-1}}$ then $\forall 1\le i\le m$, $\sum_{j=1}^{i} p_j \ge \sum_{j=1}^{i} q_j$.*

Applying these lemmas to our problem, we only need to find a probability distribution $\widehat{\pi}^-$ such that $\frac{\widehat{\pi}^-[i+1]}{\widehat{\pi}^-[i]} \le \frac{\pi(S_{i+1})}{\pi(S_i)} = \frac{\pi(S_{i+1}|S)}{\pi(S_i|S)}$. In fact, to compute $\widehat{\pi}^-$, it is sufficient to define a sequence (ρ_i) such that $\rho_1=1$ and $\forall 1\le i\le n-1,\ \rho_{i+1}\le \frac{\pi(S_{i+1})}{\pi(S_i)}$, since we can then take $\widehat{\pi}^-[i] = \frac{\rho_i\rho_{i-1}\cdots\rho_1}{\sum_{i=1}^{n}\rho_i\rho_{i-1}\cdots\rho_1}$.

4.2 Computation of the minorization ratios

In this section we provide an algorithm to compute the ratios ρ_i. Let us first set some notations: π_i (row vector with $|S_i|$ components) is the restriction of π to S_i; $\pi_{/i}$ (row vector with $|S_i|$ components) is the conditional version of π_i: $\pi_{/i}[s] = \frac{\pi[s]}{\pi(S_i)}$; $\widehat{\pi}_{/i}$ (row vector with dimension n_i) is the conditional steady state probability vector on the aggregated states $S_{i,j}$ of S_i: $\widehat{\pi}_{/i}[j] = \pi(S_{i,j}|S_i) = \frac{\sum_{s\in S_{i,j}}\pi[s]}{\pi(S_i)}$; Q is the generator of the Markov chain. Since the state space S is partitioned into disjoint subsets S_i, sequentially visited, a transition occurs only inside S_i or from S_i to S_{i+1}. We may then view Q as a block matrix $Q=[Q_{i,j}]$. $Q_{i,j}$ gives the transition rates from states of S_i to states of S_j and is a non null matrix only for $j=i,i+1$.

Principle of the algorithm We start from the set of equilibrium equations

$$\begin{aligned}
\pi_1 Q_{12} + \pi_2 Q_{22} &= 0 \\
&\vdots \\
\pi_{i-1} Q_{i-1,i} + \pi_i Q_{i,i} &= 0 \\
&\vdots \\
\pi_{n-1} Q_{n-1,n} + \pi_n Q_{n,n} &= 0
\end{aligned} \tag{6}$$

Equation (6) may be rewritten ($Q_{i,i}$ is regular since Q is a generator):

$$\pi_i = -\pi_{i-1}.Q_{i-1,i}.Q_{i,i}^{-1} \tag{7}$$

which is the basis for the iterative computation of the ρ_i.

The ith step of the algorithm computes two vectors $\widehat{\pi}^-_{/i}$ and $\widehat{\pi}^+_{/i}$ such that $\widehat{\pi}^-_{/i} \le \widehat{\pi}_{/i} \le \widehat{\pi}^+_{/i}$ and the number $\rho_i \le \frac{\pi(S_i)}{\pi(S_{i-1})}$ from the set of values previously computed:

$$\forall 1 \le j \le i-1 \begin{cases} \widehat{\pi}^-_{/j} \ \text{ and } \ \widehat{\pi}^+_{/j} & \text{with } \ \widehat{\pi}^-_{/j} \le \widehat{\pi}_{/j} \le \widehat{\pi}^+_{/j} \\ \rho_j & \text{with } \ \rho_j \le \frac{\pi(S_j)}{\pi(S_{j-1})} \end{cases}$$

Note that, although we only want to compute the ρ_i, we need to introduce the bounds $\widehat{\pi}^-_{/j}$ and $\widehat{\pi}^+_{/j}$: as we shall see, $\widehat{\pi}^-_{/i-1}$ is required to define ρ_i and $\widehat{\pi}^-_{/i-1}$ and $\widehat{\pi}^+_{/i-1}$ are required to compute $\widehat{\pi}^-_{/i}$.

Initial step Let P_1 be a "pseudo uniformized" (since $Q_{1,1}$ is not a generator) substochastic matrix associated with $Q_{1,1}$ and $M \ge \max_{s\in S}\{-q_{s,s}\}$: $P_1 = I_{|S_1|} + \frac{1}{M} Q_{1,1}$, with $I_{|S_1|}$ the $|S_1|$ Identity matrix. We choose $\widehat{P}_1^-$ ($n_1 \times n_1$ matrix) such that $\widehat{P}_1^-[x,y] \le \min_{s\in S_{1,x}}\{\sum_{s'\in S_{1,y}} P_1[s,s']\}$ and we compute an upper adapted (to P) matrix $\widehat{P}_1^+$. Like with the first method, using the visit-ratios ($\widehat{P}_1^-$ fulfills the hypothesis of lemma 1 and $\widehat{P}_1^+$ fulfills those of proposition 1 with respect to the partition $(S_{1,j})$ of S_1), we set

$$\widehat{\pi}^-_{/1}[j] = \frac{\left(\sum_{k\ge 0} (\widehat{P}_1^-)^k\right)[1,j]}{\sum_{l=1}^{n_1}\left(\sum_{k\ge 0}(\widehat{P}_1^+)^k\right)[1,l]} \quad \text{and} \quad \widehat{\pi}^+_{/1}[j] = 1 - \sum_{l\ne j} \widehat{\pi}^-_{/1}[l]$$

since $\widehat{\pi}_{/1}[j] = 1 - \sum_{l\ne j} \widehat{\pi}_{/1}[l]$.

Computations for a step i • Let us first give another expression of $-Q_{i,i}^{-1}$

If $-Q_{i,i} = [q'_{s,s'}]$ we have: $q'_{s,s} = -q_{s,s} > 0$ and $q'_{s,s'} = -q_{s,s'} < 0$ for $s \neq s'$, and $q_{s,s} \geq \sum_{s' \neq s} q_{s,s'}$, since Q is a generator. Let $M_i \geq \max_{s \in S_i} \{-q_{s,s}\}$ and P_i the substochastic $(|S_i| \times |S_i|)$ matrix such that $-Q_{i,i} = M_i\,(I - P_i)$ so that

$$P_i[s,s'] = \begin{cases} \frac{q_{s,s}}{M_i} & \text{if } \; s = s' \\ 1 + \frac{q_{s,s'}}{M_i} & \text{otherwise} \end{cases}$$

Since $Q_{i,i}$ is regular, $-Q_{i,i}^{-1} = \frac{1}{M_i}.\sum_{k \geq 0} P_i^k$ So, from equation (7)

$$\pi_i = \frac{1}{M_i}.\pi_{i-1}.Q_{i-1,i}.\sum_{k \geq 0} P_i^k = \frac{\pi(S_{i-1})}{M_i}.\pi_{/i-1}.Q_{i-1,i}.\sum_{k \geq 0} P_i^k \tag{8}$$

• Computation of $\widehat{\pi}^-_{/i}$

Since $\widehat{\pi}_{/i}[j] = \frac{1}{\pi(S_i)}\pi(S_{i,j})$, we compute a lower bound of $\pi(S_{i,j})$ and an upper bound of $\pi(S_i)$.

– Lower bound of $\pi(S_{i,j})$

From (8)

$$\pi(S_{i,j}) = \frac{1}{M_i}.\sum_{s'' \in S_{i,j}} \sum_{s' \in S_i} \sum_{s \in S_{i-1}} \pi_{i-1}[s].Q_{i-1,i}[s,s'].\left(\sum_{k \geq 0} P_i^k\right)[s',s'']$$

Moving the summation over s''

$$\pi(S_{i,j}) = \frac{1}{M_i}.\sum_{s' \in S_i} \sum_{s \in S_{i-1}} \pi_{i-1}[s].Q_{i-1,i}[s,s'].\sum_{s'' \in S_{i,j}} \left(\sum_{k \geq 0} P_i^k\right)[s',s'']$$

Splitting the summation over s' and moving it partially

$$\pi(S_{i,j}) = \frac{1}{M_i}.\sum_{v=1}^{n_i} \sum_{s \in S_{i-1}} \pi_{i-1}[s].\sum_{s' \in S_{i,v}} Q_{i-1,i}[s,s'].\sum_{s'' \in S_{i,j}} \left(\sum_{k \geq 0} P_i^k\right)[s',s'']$$

If we have a matrix $\widehat{P}_i^-$ ($n_i \times n_i$, to be computed) such that $\widehat{P}_i^-[x,y] \leq \min_{s \in S_{i,x}}$ $\{\sum_{s' \in S_{i,y}} P_i[s,s']\}$ then, from lemma 1

$$\pi(S_{i,j}) \geq \frac{1}{M_i}.\sum_{v=1}^{n_i} \sum_{s \in S_{i-1}} \pi_{i-1}[s].\sum_{s' \in S_{i,v}} Q_{i-1,i}[s,s'].\left(\sum_{k \geq 0} (\widehat{P}_i^-)^k\right)[v,j]$$

We now split the summation over s

$$\pi(S_{i,j}) \geq \frac{1}{M_i}.\sum_{v=1}^{n_i} \sum_{u=1}^{n_{i-1}} \sum_{s \in S_{i-1,u}} \pi_{i-1}[s].\sum_{s' \in S_{i,v}} Q_{i-1,i}[s,s'].\left(\sum_{k \geq 0} (\widehat{P}_i^-)^k\right)[v,j]$$

and we introduce a matrix $\widehat{Q}^-_{i-1,i}$ ($n_{i-1} \times n_i$, to be computed) such that $\widehat{Q}^-_{i-1,i}[x,y] \le \min_{s \in S_{i-1,x}} \{\sum_{s' \in S_{i,y}} Q_{i-1,i}[s,s']\}$. Hence

$$\pi(S_{i,j}) \ge \frac{1}{M_i}.\sum_{v=1}^{n_i}\sum_{u=1}^{n_{i-1}}\sum_{s \in S_{i-1,u}} \pi_{i-1}[s].\widehat{Q}^-_{i-1,i}[u,v].\left(\sum_{k \ge 0}(\widehat{P}^-_i)^k\right)[v,j]$$

from the property of the vector $\widehat{\pi}^-_{/i-1}$ we obtain

$$\pi(S_{i,j}) \ge \frac{1}{M_i}.\sum_{v=1}^{n_i}\sum_{u=1}^{n_{i-1}} \pi(S_{i-1})\widehat{\pi}^-_{/i-1}[u].\widehat{Q}^-_{i-1,i}[u,v].\left(\sum_{k \ge 0}(\widehat{P}^-_i)^k\right)[v,j]$$

that is to say

$$\pi(S_{i,j}) \ge \frac{\pi(S_{i-1})}{M_i}.\widehat{\pi}^-_{/i-1}.\widehat{Q}^-_{i-1,i}.\left(\sum_{k \ge 0}(\widehat{P}^-_i)^k\right)[j] \qquad (9)$$

– Upper bound of $\pi(S_i)$

From (8), and with the same kind of derivation

$$\pi(S_i) = \pi_i(S_i)\frac{\pi(S_{i-1})}{M_i}.\sum_{s \in S_i}\sum_{s' \in S_{i-1}} \pi_{/i-1}[s'].\sum_{s'' \in S_i} Q_{i-1,i}[s',s''].\left(\sum_{k \ge 0} P_i^k\right)[s'',s]$$

$$\pi(S_i) = \frac{\pi(S_{i-1})}{M_i}.\sum_{j=1}^{n_i}\sum_{s' \in S_{i-1}} \pi_{/i-1}[s'].\sum_{s'' \in S_i} Q_{i-1,i}[s',s''].\sum_{s \in S_{i,j}}\left(\sum_{k \ge 0} P_i^k\right)[s'',s]$$

$$\pi(S_i) = \frac{\pi(S_{i-1})}{M_i}.\sum_{j=1}^{n_i}\sum_{v=1}^{n_i}\sum_{s' \in S_{i-1}} \pi_{/i-1}[s'].\sum_{s'' \in S_{i,v}} Q_{i-1,i}[s',s''].\sum_{s \in S_{i,j}}\left(\sum_{k \ge 0} P_i^k\right)[s'',s]$$

Let $\widehat{P}^+_i$ (to be computed) be a strictly substochastic and upper bounding $n_i \times n_i$ matrix with respect to the partition $(S_{i,j})$ of S_i, then, from Proposition 1,

$$\pi(S_i) \le \frac{\pi(S_{i-1})}{M_i}.\sum_{j=1}^{n_i}\sum_{v=1}^{n_i}\sum_{s' \in S_{i-1}} \pi_{/i-1}[s'].\sum_{s'' \in S_{i,v}} Q_{i-1,i}[s',s''].\left(\sum_{k \ge 0}(\widehat{P}^+_i)^k\right)[v,j]$$

$$\pi(S_i) \le \frac{\pi(S_{i-1})}{M_i}.\sum_{j=1}^{n_i}\sum_{v=1}^{n_i}\sum_{u=1}^{n_{i-1}}\sum_{s' \in S_{i-1,u}} \pi_{/i-1}[s'].\sum_{s'' \in S_{i,v}} Q_{i-1,i}[s',s''].\left(\sum_{k \ge 0}(\widehat{P}^+_i)^k\right)[v,j]$$

Let $\widehat{Q}^+_{i-1,i}$ ($n_{i-1} \times n_i$, to be computed) be such that $\widehat{Q}^+_{i-1,i}[x,y] \ge \max_{s \in S_{i-1,x}} \{\sum_{s' \in S_{i,y}} Q_{i-1,i}[s,s']\}$. Then,

$$\pi(S_i) \le \frac{\pi(S_{i-1})}{M_i}.\sum_{j=1}^{n_i}\sum_{v=1}^{n_i}\sum_{u=1}^{n_{i-1}}\sum_{s' \in S_{i-1,u}} \pi_{/i-1}[s'].\widehat{Q}^+_{i-1,i}[u,v].\left(\sum_{k \ge 0}(\widehat{P}^+_i)^k\right)[v,j]$$

$$\pi(S_i) \leq \frac{\pi(S_{i-1})}{M_i} \cdot \sum_{j=1}^{n_i} \sum_{v=1}^{n_i} \sum_{u=1}^{n_{i-1}} \widehat{\pi}^+_{/i-1}[u].\widehat{Q}^+_{i-1,i}[u,v]. \left(\sum_{k \geq 0} (\widehat{P}^+_i)^k \right) [v,j]$$

from the property of the vector $\widehat{\pi}^+_{/i-1}$, which may rewritten as

$$\pi(S_i) \leq \frac{\pi(S_{i-1})}{M_i} . \widehat{\pi}^+_{/i-1} . \widehat{Q}^+_{i-1,i} . \left(\sum_{k \geq 0} (\widehat{P}^+_i)^k \right) .1^T_{n_i} \qquad (10)$$

with $1^T_{n_i}$ the column vector with n_i components, all equal to 1.

Finally, by (9) and (10), we set

$$\widehat{\pi}^-_{/i}[j] \stackrel{def}{=} \frac{\widehat{\pi}^-_{/i-1} . \widehat{Q}^-_{i-1,i} . \left(\sum_{k \geq 0} (\widehat{P}^-_i)^k \right) [j]}{\widehat{\pi}^+_{/i-1} . \widehat{Q}^+_{i-1,i} . \left(\sum_{k \geq 0} (\widehat{P}^+_i)^k \right) .1^T_{n_i}} \qquad (11)$$

- Computation of $\widehat{\pi}^+_{/i}$

  Since $\widehat{\pi}_{/i}[j] = 1 - \sum_{l \neq j} \widehat{\pi}_{/i}[l]$, we simply define $\widehat{\pi}^+_{/i}[j] \stackrel{def}{=} 1 - \sum_{l \neq j} \widehat{\pi}^-_{/i}[l]$.

- Computation of ρ_i

  From (9), we have $\pi(S_i) \geq \frac{\pi(S_{i-1})}{M_i} . \widehat{\pi}^-_{/i-1} . \widehat{Q}^-_{i-1,i} . \left(\sum_{k \geq 0} (\widehat{P}^-_i)^k \right) .1^T_{n_i}$ and we can define

$$\rho_i \stackrel{def}{=} \frac{1}{M_i} . \widehat{\pi}^-_{/i-1} . \widehat{Q}^-_{i-1,i} . \left(\sum_{k \geq 0} (\widehat{P}^-_i)^k \right) .1^T_{n_i} \qquad (12)$$

Note that we must have $\rho_2 \neq 0$ to obtain non trivial lower bounds $\widehat{\pi}^-[i]$. From 12, this is equivalent to the existence of j and k such that $\widehat{\pi}^-_{/1}[j] > 0$ and $\widehat{Q}^-_{1,2}[j,k] > 0$. This means that there must be a subset $S_{1,j}$ of S_1 and a subset $S_{2,k}$ of S_2 such that from each state of $S_{1,j}$ (and not of S_1 as in the first method), the system must be able to jump into $S_{2,k}$.

4.3 Example

The goal of this section is to explain how the various matrices required by the algorithm just presented may be derived from the model of the system. Numerical results are presented in the extended version [4] of the paper. We study a modified version of the example of Section 3.5 and we concentrate ourselves on the computation phase of the request service in the sub-state space S_i of S ($n - i$ requests in computation phase, $i - 1$ requests in transmission phase). This computation phase is now made up of four steps, the transmission phase being unchanged. An initial stage (rate μ_1) is followed by a fork producing two "sub-requests", with rate μ_2 and μ_3. The join of the sub-requests (rate μ_4) ends the computation phase. The distribution of the computation phase service may be viewed as a phase-type distribution of the type depicted in figure 2.

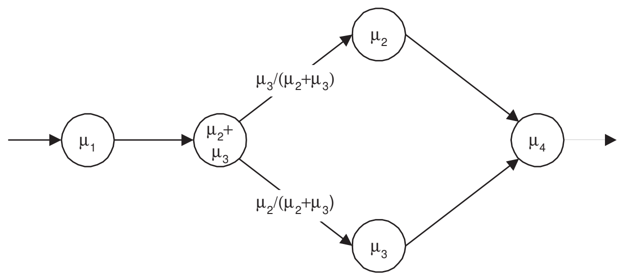

Fig. 2. Distribution of the computation phase service (method 2)

The critical choice of the method is the definition of the partitions $S_{i,j}$ for the minorization and majorization in the algorithm (the same partition was used in the presentation of the algorithm). This choice is a tradeoff between the complexity of the computations, the fulfillment of the hypothesis of the theorem 1, for the matrices $\widehat{P}_i^-$ and $\widehat{P}_i^+$, the tightness of the bounds and the information directly available from the model. Also note that the partition corresponding to the minorization must be a refinement of the one for the majorization due to the bound expressions of Section 3.3 and the computation of $\widehat{\pi}^+_{/i}$.

In our example, we may choose $S_{i,j}$ as the set of states for which j steps have been done by the whole $n-i$ requests and we can compute quantities $L_{i,j}$ and $N_{i,j}$ (the L_i and N_i values of the method 1) to define a $\widehat{\pi}^+_{/i}$ as in method 1. For what concerns the minorization, these $S_{i,j}$ are a too coarse partition, providing a weak bounding vector $\widehat{\pi}^-_{/i}$: we need more information about the advance of requests in the different steps to define a valuable lower bound. Hence we shall take $S_{i,\bar{\jmath}}$ as the partition for the minorization, where $\bar{\jmath} = (j_1, j_2, j_3, j_4)$, $\sum_{k=1}^4 j_k = n - i$ and j_k is the number of requests in kth step. It is straightforward to verify that this partition is a refinement of the majorization partition. Whatever the matrix, we may choose any value M_i such that $M_i \geq \max_{s \in S_i}\{-q_{s,s}\}$. For instance, we may set $M_i = (n-i)\max\{\mu_1, \mu_2+\mu_3, \mu_2, \mu_3, \mu_4\} + (i-1)\beta_{\max} + x_{\max} + 1$.

Minorization Here, $S_{i,\bar{\jmath}} = \{s | j_1$ requests in initial step, j_2 requests with one step done, $\ldots$, j_4 requests with three steps done$\}$.

- Recalling that we must have $\widehat{P}_i^-[x,y] \leq \min_{s \in S_{i,x}} \sum_{s' \in S_{i,y}} P_i[s,s']$, we set

$$\widehat{P}_i^-[\bar{\jmath}, \bar{\jmath}'] = \begin{cases} \frac{j_1\mu_1}{M_i} & \text{if } \bar{\jmath}' = (j_1-1, j_2+1, j_3, j_4) \\ \frac{j_2(\mu_2+\mu_3)}{M_i} & \text{if } \bar{\jmath}' = (j_1, j_2-1, j_3+1, j_4) \\ \frac{j_3 \min\{\mu_2,\mu_3\}}{M_i} & \text{if } \bar{\jmath}' = (j_1, j_2, j_3-1, j_4+1) \\ 1 - \frac{r(\bar{\jmath},\bar{\jmath}')}{M_i} & \text{if } \bar{\jmath}' = \bar{\jmath} \\ 0 & \text{otherwise} \end{cases}$$

with $r(\bar{\jmath},\bar{\jmath}') = j_1\mu_1 + j_2(\mu_2+\mu_3) + j_3\max\{\mu_2,\mu_3\} + j_4\mu_4 + (i-1)\beta_{\max} + x_{\max}$.

• We must also have $\widehat{Q}^{-}_{i-1,i}[x,y] \leq \min_{s\in S_{i-1,x}} \sum_{s'\in S_{i,y}} Q_{i-1,i}[s,s']$ so that we define $\widehat{Q}^{-}_{i-1,i}[\bar{\jmath},\bar{\jmath}'] = j_4\mu_4$ if $\bar{\jmath}' = (j_1,j_2,j_3,j_4-1)$ and 0 otherwise.

Majorization Here, $S_{i,j} = \{s|j \text{ steps have be done}\}$ $(0 \leq j \leq 3(n-i))$.

• We define $\widehat{P}^{+}_{i}$ as with method 1 (but applied to the $S_{i,j}$), which ensures that $\widehat{P}^{+}_{i}$ is a strictly substochastic and upper bounding matrix. Since we need $L_{i,j} \leq \min_{s\in S_{i,l}, l\geq j} \{Leave(s)\}$, we take $L_{i,j} = \frac{\mu_4 lnls(j)}{M_i}$, where $lnls(j)$ is a lower bound of the number of requests being in their last step of computation: $lnls(j) = \max\{0, j-2(n-i)\}$. The numbers $N_{i,j}$ must verify $0 < N_{i,j} \leq \min_{s\in S_{i,j}} \sum_{s'\in S_{i,j+1}} P[s,s']$. An accurate choice, which can be computed without overhead during the computation of $\widehat{P}^{-}_{i}$ is:
$N_{i,j} = \min_{\bar{\jmath}, \sum_{k=1}^{4}(k-1)j_k = j} \frac{\mu_1 j_1 + (\mu_2+\mu_3)j_2 + \min\{\mu_2,\mu_3\}j_3}{M_i}$.

• $\widehat{Q}^{+}_{i-1,i}$ must fulfills $\widehat{Q}^{+}_{i-1,i}[j,j'] \geq \max_{s\in S_{i-1,j}} \sum_{s'\in S_{i,j'}} Q_{i-1,i}[s,s']$, hence we define $\widehat{Q}^{+}_{i-1,i}[j,j'] = \mu_4 unls(j)$ if $j' = j-3$ and 0 otherwise. $unls(j)$ is an upper bound of the number of requests being in their last step of computation: $unls(j) = j \operatorname{div} 3$.

5 Conclusion

In this paper we have presented a new approach to compute bounds of performance measures of Markov chains frequently encountered in system modelling. This approach provides important savings in computation and memory requirements with respect to an exact computation of the steady state distribution. When a subset of the state space of the chain may be partitioned in subspaces sequentially visited, we have defined bounds of conditional steady state reward rates on theses subspaces or on the whole subset. Two methods have been proposed, corresponding to different properties of the model. Our methods are based on a combination of strong stochastic ordering and aggregation. We have explained how to compute these bounds and reported first experiments showing the interest of the approach when the system fulfills suited hypothesis. Work is in progress to evaluate accurately the behaviour of the results for the second, more elaborated, method. We are also studying the applicability of our method to domains usually involving bounding methods, like performance evaluation of fault tolerant and repairable systems, for which the reparation rates may be seen, themselves, as rewards. In particular, we are working on application of our results to systems with multiple entry points in the subsets visited.

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