Ordinal Theory for Expressiveness of Well Structured Transition Systems

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Abstract. To the best of our knowledge, we characterize for the first time the importance of resources (counters, channels, alphabets) when measuring expressiveness of WSTS. We establish, for usual classes of wpos, the equivalence between the existence of order reflections (non-monotonic order embeddings) and the simulations with respect to coverability languages. We show that the non-existence of order reflections can be proved by the computation of order types. This allows us to solve some open problems and to unify the existing proofs of the WSTS classification.

1 Introduction

WSTS. Infinite-state systems appear in a lot of models and applications: stack automata, counter systems, Petri nets or VASSs, reset/transfer Petri nets, fifo (lossy) channel systems, parameterized systems. Among these infinite-state systems, a part of them, called Well-Structured Transition Systems (WSTS) [7] enjoys two nice properties: there is a well partial ordering (wpo) on the set of states and the transition relation is monotone with respect to this wpo.

The theory of WSTS has been successfully applied for the verification of safety properties of numerous infinite-state models like Lossy Channel Systems, extensions of Petri Nets like reset/transfer and Affine Well Nets [8], or broadcast protocols. Most of the positive results are based on the decidability of the coverability problem (whether an upward closed set of states is reachable from the initial state) for WSTS, under natural effectiveness hypotheses. The reachability problem, on the contrary, is undecidable even for the class of Petri nets extended with reset or transfer transitions.

Expressiveness. Well Structured Languages [9] were introduced as a measure of the expressiveness of subclasses of WSTS. More precisely, the language of an

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instance of a model is defined as the class of *finite* words accepted by it, with *coverability* as accepting condition, that is, generated by traces that reach a state which is bigger than a given final state. Convincing arguments show that the class of coverability languages is the right one. For instance, though reachability languages are more precise than coverability languages, the class of reachability languages is RE for almost all Petri Nets extensions containing Reset Petri Nets or Transfer Petri Nets.

The expressive power of WSTS comes from two natural sources: from the structure of the state space and from the semantics of the transition relation. These two notions were often extremely interwined in the proofs. We propose ourselves to separate them in order to have a formal and generalizable method.

The study of the state space is related to the relevance of resources: A natural question when confronted to an extension of a model is whether the additional resources actually yield an increase in expressiveness. For example, if we look at Timed Automata, clocks are a strict resource: Timed Automata with k clocks are less expressive that Timed Automata with $k+1$ clocks [4]. Surprisingly, no similar results exist for well-known models like Petri Nets (with respect to the number of places) or Lossy Channel Systems (with respect to the number of channels, or number of symbols in the alphabet) except in some particular recent works [6].

Ordinal theory for partial orders. Ordinals are a well-known representation of well-founded total orders. Thanks to de Jongh, Parikh, Schmidt ([10], [16]) and others, this representation has been extended to well partial orders. We are mainly interested in the order type of a wpo, which can be understood as the "size" of the order. The order types of the union, product, and finite words have been computed since de Jongh and Parikh. Recently, Weiermann [17] has completed this view by computing the order type for multisets.

Contribution. First, we introduce order reflections, a variation of order embeddings that are allowed to be non-monotonic. We define a notion of witnessing, that reflects the ability of a WSTS to recognize a wpo through a coverability language. We establish the equivalence between the existence of order reflections and the simulations with respect to coverability languages, modulo the ability of the WSTS classes to witness their own state space.

Second, we show how to use results from the theory of ordinals, and more precisely the properties of maximal order types, studied by de Jongh and Parikh [10] and Schmidt [16] to easily prove the absence of reflections.

Last, we study Lossy Channel Systems and extensions of Petri Nets. We show that most of known classes of WSTS are self-witnessing. This allows us to unify and simplify the existing proofs regarding the classification of WSTS, also solving the open problem [14] of the relative expressiveness of two Petri Nets extensions called ν-Petri Nets and Data Nets, also yielding that the number of unbounded places for these Petri Nets extensions and the size of the alphabet for Lossy Channel Systems are relevant resources when considering their expressiveness.

Related work. Coverability languages have been used to discriminate the expressive power of several WSTS, like Lossy Channel Systems or several monotonic extensions of Petri Nets. In [9] several pumping lemmas are proved to discriminate between extensions of Petri Nets. In [1, 2] the expressive power of Petri Nets is proved to be strictly below that of Affine Well Nets, and Affine Well Nets are proved to be strictly less expressive than Lossy Channel Systems. Similar results are obtained in [14], though some significant problems are left open, like the distinction between ν-Petri Nets [13] and Data Nets [12] that we solve here.

Outline. The rest of the paper is organized as follows. In Section 2 we introduce wpos, WSTS and ordinals. Then in Section 3 we develop the study of reflections and its links with expressiveness of WSTS. Afterwards in Section 4 we apply our result to the classical models of Petri Nets and Lossy Channel Systems. Section 5 presents the extension of our results applicable to more recent models of WSTS. Finally we conclude and give perspectives to this work in Section 6. Appendices that will be omitted in the final version include the complete proofs of our results.

2 Preliminaries and WSTS

Well Orders. $(X, \leq_X)$ is a *quasi-order* (qo) if $\leq_X$ is a reflexive and transitive binary relation on X. For a qo we write $x <_X y$ iff $x \leq_X y$ and $y \not\leq_X x$. A *partial order* (po) is an antisymmetric quasi-order. Given any qo $(X, \leq_X)$, the quotient set $X/\equiv_{\leq_X}$ is a po where $x \equiv_{\leq_X} y$ is defined by $x \leq_X y \wedge y \leq_X x$. Hence, in all the paper, we will suppose that $(X, \leq_X)$ is a po.

The *downward closure* of a subset $A \subseteq X$ is defined as $\downarrow A = \{x \in X \mid \exists x' \in A,\ x \leq x'\}$. A subset A is *downward closed* iff $\downarrow A = A$. A po $(X, \leq_X)$ is a *well partial order* (wpo) if for every infinite sequence $x_0, x_1, \ldots \in X$ there are i and j with $i < j$ such that $x_i \leq x_j$. Equivalently, a po is a wpo when there are no strictly decreasing (for inclusion) sequences of downward closed sets.

We will shorten $(X, \leq_X)$ to X when the underlying order is obvious. Similarly, $\leq$ will be used instead of $\leq_X$ when X can be deduced from the context.

If X and Y are wpos, their cartesian product, denoted $X \times Y$ is well ordered by $(x, y) \leq_{X\times Y} (x', y') \iff x \leq_X x' \wedge y \leq_Y y'$. Their disjoint union, denoted $X \uplus Y$ is well ordered by:

$$z \leq_{X \uplus Y} z' \iff \begin{cases} z, z' \in X \\ z \leq_X z' \end{cases} \text{or} \quad \begin{cases} z, z' \in Y \\ z \leq_Y z' \end{cases}$$

A po $(X, \leq)$ is *total* (or *linear*) if for any $x, x' \in X$ either $x \leq x'$ or $x' \leq x$. If $(X_i, \leq_i)$ are total po for $i \in \mathbb{N}$ we can define the (irreflexive) total order $<_{lex}$ in $\bigcup_k X_1 \times ... \times X_k$ by $(x_1, ..., x_p) <_{lex} (x'_1, ..., x'_q)$ iff there is $i \in \{1, ..., min(p, q)\}$ such that $x_j = x'_j$ for $j < i$ and $x_i <_i x'_i$ or $(x_1, ..., x_p) = (x'_1, ..., x'_p)$ and $q > p$. Then $\leq_{lex}$ given by $x \leq_{lex} x'$ iff $x = x'$ or $x <_{lex} x'$ is a total order.

Functions. Given a partial function (shortly: function) $f : X \to Y$, the *domain* of f is defined by $dom(f) = \{x \in X \mid \exists y \in Y,\ f(x) = y\}$ and its *range* by $range(f) = \{y \in Y \mid \exists x \in X,\ f(x) = y\}$. A function f is *surjective* if $range(f) = Y$ and it is *total* if $dom(f) = X$. Total functions are called *mappings*.

A mapping f is *injective* if for all x, x', $f(x) = f(x') \implies x = x'$. Finally, let us consider a mapping f: if X and Y are ordered, f is *increasing* (resp. *strictly increasing*) if $x \leq_X y \implies f(x) \leq_Y f(y)$ (resp. if $x <_X y \implies f(x) <_Y f(y)$); f is an *order embedding* (shortly: embedding) if $f(x) \leq_Y f(x') \iff x \leq_X x'$. A bijective order embedding is called an *order isomorphism* (shortly: isomorphism).

Multisets. Given a set X, we denote by $X^{\oplus}$ the set of finite multisets of X, that is, the set of mappings $m : X \to \mathbb{N}$ with a finite support $sup(m) = \{x \in X \mid m(x) \neq 0\}$. We use the set-like notation $\{\!|...|\!\}$ for multisets when convenient, with $\{\!|x^n|\!\}$ describing the multiset containing x n times. We use $+$ and $-$ for multiset operations. If X is a wpo then so is $X^{\oplus}$ ordered by $\leq_{\oplus}$ defined by $\{\!|x_1, \ldots, x_n|\!\} \leq_{\oplus} \{\!|x'_1, \ldots, x'_m|\!\}$ if there is an injection $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $x_i \leq_X x'_{h(i)}$ for each $i \in \{1, \ldots, n\}$.

Words. Given a set X, any $u = x_1 \cdots x_n$ with $n \geq 0$ and $x_i \in X$, for all i, is a finite word on X. We denote by X^* the set of finite words on X. If $n = 0$ then u is the empty word, which is denoted by ϵ. A language L on X is a subset of X^*. Given L and L' two languages on X^*, we define the language $LL' = \{uv \mid u \in L, v \in L'\}$. If X is a wpo then so is X^* ordered by $\leq_{X^*}$ which is defined as follows: $x_1 \ldots x_n \leq_{X^*} x'_1 \ldots x'_m$ if there is a strictly increasing mapping $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $x_i \leq_X x'_{h(i)}$ for each $i \in \{1, \ldots, n\}$.

Ordinals below ϵ_0. In this paper, we shall work with set theoretical ordinals. Let us recall a few properties of these objects. The class of ordinals is totally ordered by inclusion, and each ordinal α is equal to the set of ordinals $\{\beta \mid \beta < \alpha\}$ below it. Every total well order $(X, \leq_X)$ is isomorphic to a unique ordinal $ot(X, \leq_X)$, called the *order type* of X.

In the context of ordinals, we define $0 = \emptyset$, $n = \{0, ..., n-1\}$ and $\omega = \mathbb{N}$, ordered by the usual order. Moreover, given α and α' ordinals, we define $\alpha + \alpha'$ as the order type of $(\{0\} \times \alpha) \cup (\{1\} \times \alpha')$ ordered by $\leq_{lex}$. In the same way, $\alpha * \alpha'$ is defined as the order type of $\alpha' \times \alpha$ ordered by $\leq_{lex}$. Note that these operations are not commutative: we have $1 + \omega = \omega \neq \omega + 1$. This definition of $+$ and $*$ coincides with the usual operations on $\mathbb{N}$ for ordinals below ω and we have $\alpha + \overset{k}{\cdots} + \alpha = \alpha * k$. Exponentiation can be similarly defined, but for simplicity of presentation, we let this definition outside this short introduction to ordinals. Note that the most important properties of exponentiation can be obtained from the ordering on Cantor's Normal Forms (CNF) that we develop below.

In this paper, we will work with ordinals below ϵ_0, that is, those that can be bounded by a tower $\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}$. These can be represented by the hierarchy of ordinals in CNF that is recursively given by the following rules:

$C_0 = \{0\}$.
$C_{n+1} = \left\{\omega^{\alpha_1} + \cdots + \omega^{\alpha_p} \mid p \in \mathbb{N},\ \alpha_1, \ldots, \alpha_p \in C_n \text{ and } \alpha_1 \geq \cdots \geq \alpha_p\right\}$ ordered by

$$\omega^{\alpha_1} + \cdots + \omega^{\alpha_p} \leq \omega^{\alpha'_1} + \cdots + \omega^{\alpha'_q} \iff (\alpha_1, \ldots, \alpha_p) \leq_{lex} (\alpha'_1, \ldots, \alpha'_q)$$

Each ordinal below ϵ_0 has a unique CNF. If $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, we denote by $Cantor(\alpha)$ the multiset $\{\!|\beta_1, \ldots, \beta_n|\!\}$.

WSTS. A *Labelled Transition System* (lts) is a tuple $\mathcal{S} = \langle X, \Sigma, \rightarrow \rangle$ where X is the set of states, Σ is the labelling alphabet and $\rightarrow \subseteq X \times (\Sigma \cup \{\epsilon\}) \times X$ is the transition relation. We write $x \xrightarrow{a} x'$ to say that $(x, a, x') \in \rightarrow$. This relation is extended for $u \in \Sigma^*$ by $x \xrightarrow{u} x' \iff x \xrightarrow{a_1} x_1 ... x_{k-1} \xrightarrow{a_k} x'$ and $u = a_1 a_2 \cdots a_k$ (note that some a_i's can be ϵ). A *Well Structured Transition System* (shortly a WSTS) is a tuple $\mathcal{S} = (X, \Sigma, \rightarrow, \leq)$, where $(X, \Sigma, \rightarrow)$ is an lts, and $\leq$ is a wpo on X, satisfying the following monotonicity condition: for all $x_1, x_2, x'_1 \in X, u \in \Sigma^*$, $x_1 \leq x'_1$, $x_1 \xrightarrow{u} x_2$ implies the existence of $x'_2 \in X$ such that $x'_1 \xrightarrow{u} x'_2$ and $x_2 \leq x'_2$. For a class $\mathbf{X}$ of wpos, we will denote by $WSTS_{\mathbf{X}}$ the class of WSTS with state space in $\mathbf{X}$, or just $WSTS_X$ for $WSTS_{\{X\}}$.

Coverability and Reachability Languages. Trace languages, reachability languages and coverability languages are natural candidates for measuring the expressive power of classes of WSTS. Given a WSTS $\mathcal{S}$ and two states x_0 and x_f, the reachability language is $L_R(\mathcal{S}, x_0, x_f) = \{u \in \Sigma^* \mid x_0 \xrightarrow{u} x_f\}$ while the coverability language is $L(\mathcal{S}, x_0, x_f) = \{u \in \Sigma^* \mid x_0 \xrightarrow{u} x,\ x \geq x_f\}$. Let us remark that all trace languages are coverability languages in taking $x_f = \perp$ where $\perp$ is the least element of X.

The class of reachability languages is the set of recursively enumerable languages for almost all Petri nets extensions containing reset Petri nets or transfer Petri nets. Thus such a criterium does not discriminates sufficiently . One could consider infinite coverability languages. A sensible accepting condition in this case could be repeated coverability, that is, the capacity of covering a given marking infinitely often, in the style of Büchi automata. However, analogously to what happens with reachability, repeated coverability is generally undecidable, which makes ω-languages a bad candidate to study the relative expressive power of WSTS. In conclusion, we will use the class of coverability languages, as in [9, 1, 2, 14]

For two classes of WSTS, $\mathbf{S}_1$ and $\mathbf{S}_2$, we write $\mathbf{S}_1 \preceq \mathbf{S}_2$ whenever for every language $L(\mathcal{S}_1, x_1, x'_1)$ with $\mathcal{S}_1 \in \mathbf{S}_1$, and x_1, x'_1 two states of $\mathcal{S}_1$, there exists another system $\mathcal{S}_2 \in \mathbf{S}_2$ and two states x_2, x'_2 of $\mathcal{S}_2$ such that $L(\mathcal{S}_2, x_2, x'_2) = L(\mathcal{S}_1, x_1, x'_1)$. When $\mathbf{S}_1 \preceq \mathbf{S}_2$ and $\mathbf{S}_2 \preceq \mathbf{S}_1$, one denotes the equivalence of classes by $\mathbf{S}_1 \simeq \mathbf{S}_2$. We write $\mathbf{S}_1 \prec \mathbf{S}_2$ for $\mathbf{S}_1 \preceq \mathbf{S}_2$ and $\mathbf{S}_2 \not\preceq \mathbf{S}_1$.

The Lossy semantics. The *lossy* semantics $\mathcal{S}_l$ of a WSTS $\mathcal{S}$ with space X is the original system $\mathcal{S}$ completed by all ϵ-transitions $x \xrightarrow{\epsilon} y$, for all $x, y \in X$ such that $y < x$. We observe that $\mathcal{S}_l$ satisfies the monotonicity condition, hence $\mathcal{S}_l$ is still a WSTS; and moreover, due to the lossy semantics, one has: for all $x_1, x_2 \in X, u \in \Sigma^*$, $x_1 \xrightarrow{u} x_2$ implies $x_1 \xrightarrow{u} x'_2$ for all $x'_2 \leq x_2$. For any x_0, x_f, we have: $L(\mathcal{S}, x_0, x_f) = L(\mathcal{S}_l, x_0, x_f)$

3 A method for comparing WSTS

In this section we propose a method to compare the expressiveness of WSTS mainly based on their state space. We will prove some results that will provide us with tools to establish strict relations between classes of WSTS.

3.1 A new tool: order reflections

Definition 1. *Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two partially ordered sets. A mapping $\varphi : X \to Y$ is an* order reflection *(shortly: reflection) if $\varphi(x) \leq_Y \varphi(x')$ implies $x \leq_X x'$ for all $x, x' \in X$.*

We will write $X \sqsubseteq Y$ if there is an embedding from X to Y and $X \sqsubseteq_{refl} Y$ if there is a reflection from X to Y. We will use $\not\sqsubseteq$ and $\not\sqsubseteq_{refl}$ for their negation and $\sqsubset$ and $\sqsubset_{refl}$ for their antisymmetric version (i.e. $X \sqsubset Y \iff X \sqsubseteq Y \wedge Y \not\sqsubseteq X$). Here are some basic properties of reflections we will use throughout the paper: for any set X, any injective mapping to $(X, =)$ is a reflection; every reflection is injective; the composition of two reflections is a reflection (so $\sqsubseteq_{refl}$ is a qo).

Furthermore, if φ is an embedding from X to Y then X is isomorphic to $\varphi(X)$ and hence can be identified to it. Clearly, existence of embeddings are a stronger requirement than the existence of reflections. In particular, it can be the case that a wpo X cannot be embedded in another wpo Y, even if there are reflections from X to Y, as implied by the following result.

Proposition 1. *The following properties hold:*
- $\mathbb{N}^k \sqsubseteq_{refl} \mathbb{N}^{\oplus}$*, for any* $k > 0$.
- $\mathbb{N}^k \not\sqsubseteq \mathbb{N}^{\oplus}$ *for any* $k \geq 3$ *(but* $\mathbb{N}^2 \sqsubseteq \mathbb{N}^{\oplus}$*).*

Proof. See Appendix A, propositions 16, 17 and 18.

3.2 Expressiveness of WSTS and order reflections

Reflections are more appropriate than embeddings for the comparison of WSTS. In particular, the existence of a reflection implies the relation between the corresponding classes of WSTS.

Theorem 1. *Let X and Y be two wpo. We have:*

$$X \sqsubseteq_{refl} Y \implies \mathit{WSTS}_X \preceq \mathit{WSTS}_Y$$

This is easily shown by taking a WSTS of state space X, looking at its lossy equivalent through the order reflection, and realizing this is another WSTS which recognizes the same language. The detailed proof is in Appendix A.

We would like to obtain the converse of the previous result: $X \not\sqsubseteq_{refl} Y \implies \mathit{WSTS}_X \not\preceq \mathit{WSTS}_Y$. First, we only present this result for "simple" state spaces. The case of more complex state spaces will be handled in later sections.

Given an alphabet $\Sigma = \{a_1, ..., a_k\}$, we define $\overline{\Sigma}$ by $\overline{\Sigma} = \{\overline{a_1}, \cdots, \overline{a_k}\}$ where $\overline{a_i}$'s are fresh symbols (i.e. $\Sigma \cap \overline{\Sigma} = \emptyset$). This notation is extended to words by $\overline{u} = \overline{a_1} \cdots \overline{a_k}$ for $u = a_1 \cdots a_k \in \Sigma^*$. In the same way, given $L \subseteq \Sigma^*$, we have $\overline{L} = \{\overline{u} \mid u \in L\} \subseteq \overline{\Sigma}^*$.

Definition 2. *Let X be a wpo and Σ a finite alphabet. A surjective partial function from Σ^* to X is called a Σ-representation of X. Given a Σ-representation γ of X, we define $L_\gamma = \{u\bar{v} \mid u, v \in dom(\gamma)$ and $\gamma(v) \leq \gamma(u)\}$. A language $L \in (\Sigma \cup \bar{\Sigma})^*$ is a γ-witness (shortly: witness) of X if $L \cap dom(\gamma)\overline{dom(\gamma)} = L_\gamma$.*

In particular, L_γ is a witness of X for any Σ-representation γ of X. Intuitively, given a witness L of X, the fact that a WSTS can recognize L *witnesses* that the WSTS can represent the structure of X: it is capable of accepting all words starting with some u (representing some state $\gamma(u)$), followed by some v that represents $\gamma(v) \leq \gamma(u)$. Witness languages are useful in proving strict relations between classes of WSTS:

Theorem 2. *Let L be a witness of X. If $X \not\sqsubseteq_{refl} Y$ then there are no $y_0, y_f \in Y$ and no $\mathcal{S} \in WSTS_Y$ such that $L = L(\mathcal{S}, y_0, y_f)$.*

Proof. Assume by contradiction that L is a covering language of a WSTS $\mathcal{S}$ whose state space is Y with y_0 and y_f as initial and final states, respectively. For each $x \in X$, let us take $u_x \in \Sigma^*$ such that $\gamma(u_x) = x$. The word $u_x\overline{u_x}$ is recognized by $\mathcal{S}$, hence we can find y_x and y'_x such that $y_0 \xrightarrow{u_x} y_x \xrightarrow{\overline{u_x}} y'_x \geq y_f$.

We define $\varphi(x) = y_x$. Let us see that φ is an order reflection from X to Y, thus reaching a contradiction. Assume that $\varphi(x) \leq \varphi(x')$. Since $\mathcal{S}$ is a WSTS any sequence fireable from $\varphi(x)$ is also fireable from $\varphi(x')$ and the state reached by this subsequence is greater or equal than the one reached from $\varphi(x)$. Hence, the state reached after $u_{x'}\overline{u_x}$ is bigger than the one reached after $u_x\overline{u_x}$, which means that $u_{x'}\overline{u_x} \in L \cap dom(\gamma)\overline{dom(\gamma)}$, implying $x \leq x'$, so that φ is an order reflection.

The simple state spaces we mentioned before, will be the ones produced by the following grammar:

$\Gamma ::= Q$	(finite set with equality)
$\mid\ \mathbb{N}$	(naturals with the standard order)
$\mid\ \Sigma^*$	(words on a finite set with the order defined in Section 2)
$\mid\ \Gamma \times \Gamma$	(cartesian product with the order defined in Section 2)

As $\mathbb{N}$ is isomorphic to Σ^* when Σ is a singleton, any set produced by Γ is isomorphic to a set $Q \times \Sigma_1^* \times \cdots \times \Sigma_k^*$ where Q and each Σ_i are finite sets.

Proposition 2. *Let X be a set produced by the grammar Γ. Then, there is a witness of X that is recognized by a WSTS of state space X.*

When a WSTS can recognize a witness of its own state space the following holds:

Proposition 3. *Let X be a wpo produced by Γ and Y any wpo. Then,*

$$X \sqsubseteq_{refl} Y \iff WSTS_X \preceq WSTS_Y$$

Proof. The direction from left to right is given by Theorem 1. For the converse, let us prove that $X \not\sqsubseteq_{refl} Y \Rightarrow WSTS_X \not\preceq WSTS_Y$. We can find a witness L of X recognized by a WSTS of state space X (Prop. 2). By Theorem 2, this language can not be recognized by a WSTS of state space Y, hence the result.

3.3 Self-witnessing WSTS classes

The reason we were able to build our equivalence between the existence of a reflection from X to Y and $WSTS_X \preceq WSTS_Y$ for any wpo X produced by Γ was Prop. 2. However, we conjecture that for any state space X that embeds $\mathbb{N}^{\oplus}$, there is no WSTS of state space X that can recognize a witness of X. This prompts us to define a new notion:

Definition 3. *Let* $\mathbf{X}$ *be a class of wpos and* $\mathbf{S}$ *a class of WSTS whose state spaces are included in* $\mathbf{X}$*.* $(\mathbf{X}, \mathbf{S})$ *is self-witnessing if, for all* $X \in \mathbf{X}$*, there exists* $\mathcal{S} \in \mathbf{S}$ *that recognizes a witness of* X*.*

We will shorten $(\mathbf{X}, \mathbf{S})$ as $\mathbf{S}$ when the state space is not explicitly needed. We extend the relation $\sqsubseteq_{refl}$ to classes of wpo by $\mathbf{X} \sqsubseteq_{refl} \mathbf{X}'$ if for any $X \in \mathbf{X}$, there exists $X' \in \mathbf{X}'$ such that $X \sqsubseteq_{refl} X'$.

Proposition 4. *Let* $(\mathbf{X}, \mathbf{S})$ *be a self-witnessing WSTS class and* $\mathbf{S}'$ *a WSTS class using state spaces inside* $\mathbf{X}'$*. Then,* $\mathbf{S} \preceq \mathbf{S}' \implies \mathbf{X} \sqsubseteq_{refl} \mathbf{X}'$*.*
Moreover, if $\mathbf{S}' = WSTS_{\mathbf{X}'}$*,* $\mathbf{S} \preceq \mathbf{S}' \iff \mathbf{X} \sqsubseteq_{refl} \mathbf{X}'$*.*

Proof. Let us show the first implication. Let $X \in \mathbf{X}$. Since $(\mathbf{X}, \mathbf{S})$ is self-witnessing, there is $\mathcal{S} \in \mathbf{S}$ that recognizes L, a witness of X. Because $\mathbf{S} \preceq \mathbf{S}'$, there is $\mathcal{S}' \in \mathbf{S}'$ recognizing L. $\mathcal{S}'$ has state space $X' \in \mathbf{X}'$, and by Theorem 2, $X \sqsubseteq_{refl} X'$.

For the second implication, for any $X \in \mathbf{X}$, we have $X' \in \mathbf{X}'$ such that $X \sqsubseteq_{refl} X'$. Because of Theorem 1, $WSTS_X \preceq WSTS_{X'}$. Hence, $WSTS_{\mathbf{X}} \preceq WSTS_{\mathbf{X}'}$.

We will see in sections 4 and 5 that many usual classes of WSTS, even those outside the algebra Γ, are self-witnessing.

3.4 How to prove the non-existence of reflections?

Because of Prop. 3 and Prop. 4, the non existence of reflections will be a powerful tool to prove strict relations between WSTS. We provide here a simple way from order theory. Let us recall that a *linearization* of a po $\leq_X$ is a linear order $\leq'_X$ on X such that $x \leq_X y \implies x \leq'_X y$. A linearization of a wpo is a well total order, hence isomorphic to an ordinal. We extend the definition of order types to non-total wpos:

Definition 4. *Let* $(X, \leq_X)$ *be a wpo. The* maximal order type *(shortly: order type) of* $(X, \leq_X)$ *is* $ot(X, \leq_X) = sup\ \{ot(X, \leq'_X) \mid \leq'_X$ *linearization of* $\leq_X\}$*.*

The existence of the *sup* comes from ordinal theory. de Jongh and Parikh [10] even show that this *sup* is actually attained. Let $Down(X)$ be the set of downward closed subsets of X. Then, another well-known characterization of the maximal order type is the following (proofs of propositions 5 and 6 are in Appendix A):

Proposition 5. $ot(X)+1 = sup\ \{\alpha \mid \exists f : \alpha \to Down(X),\ f \text{ strictly increasing}\}$

This leads us to the proposition that we use to separate many classes of WSTS:

Proposition 6. *[17] Let X and Y be two wpos.* $X \sqsubseteq_{refl} Y \implies ot(X) \leq ot(Y)$.

The order types of the usual state spaces used for WSTS are known. We will recall some classic results on these order types, but we need the following definitions of addition and multiplication on ordinals to be able to characterize the order types of $X \uplus Y$ and $X \times Y$. Remember (Section 2) that an ordinal α below ε_0 is uniquely determined by $Cantor(\alpha)$, hence the validity of the following definition.

Definition 5. *(Hessenberg 1906, [10]) The natural addition, denoted $\oplus$, and the natural multiplication, denoted $\otimes$, are defined by:*

$$Cantor(\alpha \oplus \alpha') = Cantor(\alpha) + Cantor(\alpha')$$
$$Cantor(\alpha \otimes \alpha') = \{\!|\beta \oplus \beta' \mid \beta \in Cantor(\alpha), \beta' \in Cantor(\alpha')|\!\}$$

We already know that the order type of a finite set (with any order) is its cardinality and that the order type of $\mathbb{N}$ is ω. De Jongh and Parikh [10], and Schmidt [16] have shown a way to compose order types with the disjoint union, the cartesian product, and the Higman ordering. A more recent and difficult result, by Weiermann [17], provides us with the order type of multisets. These results are summed up here:

Proposition 7. *([10], [16], [17])*

- $ot(X \uplus Y) = ot(X) \oplus ot(Y)$
- $ot(X \times Y) = ot(X) \otimes ot(Y)$
- $ot(X^*) = \begin{cases} \omega^{\omega^{ot(X)-1}} & \text{if } X \text{ finite} \\ \omega^{\omega^{ot(X)}} & \text{otherwise (for } ot(X) < \epsilon_0) \end{cases}$
- $ot(X^{\oplus}) = \omega^{ot(X)}$ *for* $ot(X) < \epsilon_0$

Formulas exist even for $ot(X) \geq \epsilon_0$. We refer the interested reader to [10] and [17] for the complete formulas. With these general results we can obtain many strict relations between wpo.

Corollary 1. *The following strict relations hold for any $k > 0$:*

(1) $\mathbb{N}^k \sqsubset_{refl} \mathbb{N}^{k+1}$
(2) $(\mathbb{N}^k)^{\oplus} \sqsubset_{refl} (\mathbb{N}^{k+1})^{\oplus}$
(3) $(\mathbb{N}^k)^* \sqsubset_{refl} (\mathbb{N}^{k+1})^*$
(4) $\mathbb{N}^k \sqsubset_{refl} \mathbb{N}^{\oplus}$
(5) $\mathbb{N}^k \sqsubset_{refl} \Sigma^*$ *(for $|\Sigma| > 1$)*

Proof. The non-strict relations in (1), (2) and (3) are clear, and for (4) this is Prop. 1. For (5), $\varphi(n_1, \ldots, n_k) = a^{n_1}b \ldots ba^{n_k}$ is a reflection. Strictness follows from Prop. 6 and the following order types, obtained according to the previous results: $ot(\mathbb{N}^k) = \omega^k$, $ot((\mathbb{N}^k)^{\oplus}) = \omega^{\omega^k}$, $ot((\mathbb{N}^k)^*) = \omega^{\omega^{\omega^k}}$, and $ot(\Sigma^*) = \omega^{\omega^{|\Sigma|-1}}$.

4 Vector Addition Systems and Lossy Channel Systems

The state spaces described by Prop. 3 are exactly those of Petri Nets and Lossy Channel Systems. We will look more closely at these systems to see the implication of this theorem regarding their expressiveness.

4.1 Vector Addition Systems and Petri nets

We work with *Vector Addition Systems with States* (VASS), which are equivalent to Petri nets. A VASS of dimension k is a tuple $(Q, T, \delta, \Sigma, \lambda)$, where Q is a finite (and non-empty) set of control sates, T is a finite set of transitions, $\delta : T \rightarrow Q \times \mathbb{Z}^k \times Q$, Σ is the finite labelling alphabet, and $\lambda : T \rightarrow \Sigma \cup \{\epsilon\}$ is the mapping which labels transitions. Transition t is enabled in (p, x) if $\delta(t) = (p, y, q)$ for some $q \in Q$ and some $y \in \mathbb{Z}^k$ with $x \geq -y$, in which case t can occur, reaching state $(q, x + y)$. VASS are WSTS by taking $(p, x) \leq (q, y)$ iff $p = q$ and $x \leq y$. The transition relation $\rightarrow$ of the WSTS associated with the VASS is defined by: $((p, x), a, (q, x + y)) \in\rightarrow$ if there is a transition $t \in T$ which is enabled in (p, x) such that $\delta(t) = (p, y, q)$ and $\lambda(t) = a$.

Let us denote by VASS_k the class of VASS with dimension k. Notice that the state space of any VASS with dimension k is in $\mathbf{X}_k = \{Q \times \mathbb{N}^k \mid Q \text{ finite}\}$. Then we have the following:

Theorem 3. *For any $k > 0$, $\mathit{VASS}_k \not\preceq \mathit{WSTS}_{\mathbf{X}_{k-1}}$.*

Proof. We remark that the WSTS defined in the proof of Prop. 2 is actually a lossy VASS when $X = Q \times \mathbb{N}^k$. This induces that we can take the non-lossy version of this VASS, which is still a WSTS. Hence, VASS_k is self-witnessing, and therefore so is $\mathit{WSTS}_{\mathbf{X}_k}$. Since $\mathbb{N}^k \not\sqsubseteq_{refl} Q \times \mathbb{N}^{k-1}$ for all finite Q (indeed, $ot(\mathbb{N}^k) = \omega^k \not\leq \omega^{k-1} * |Q| = ot(Q \times \mathbb{N}^{k-1})$), we have $\mathbf{X}_k \not\sqsubseteq_{refl} \mathbf{X}_{k-1}$ and by Prop. 4 we conclude.

We remark that even the class of lossy VASS with dimension k is not included in the class of WSTS with state space in $\mathbf{X}_{k-1}$. Moreover, if we consider *Affine Well Nets* (*AWN*) (an extension of Petri nets with whole-place operations like transfers or resets), and denote by AWN_k the class of AWN with k unbounded places (therefeore, with state space in $\mathbf{X}_k$), we can obtain from the previous result the following simple consequences.

Corollary 2. *$\mathit{VASS}_k \prec \mathit{VASS}_{k+1} \not\preceq \mathit{AWN}_k$ for all $k \geq 0$.*

4.2 Lossy Channel Systems

Let Op denote any vector of k operations on a (fifo) channel such that for every $i \in \{1, \ldots, k\}$, $Op(i)$ is either a send operation $!a$ on channel i, a receive operation $?a$ from channel i ($a \in A$), a test for emptyness $\epsilon?$ on channel i or a null operation *nop*. Let us denote OP_k the set of operations Op.

A *Lossy Channel System* (LCS)[3] with k channels is a tuple $(Q, A, T, \delta, \Sigma, \lambda)$ where Q is a finite (and non-empty) set of states, A is the finite set of messages, T is a finite set of transitions, $\delta : T \to Q \times OP_k \times Q$, Σ is the labelling alphabet and $\lambda : T \to \Sigma \cup \{\epsilon\}$ is the mapping which labels transitions. The set of configurations is $Q \times (A^*)^k$.

For (non lossy) channel systems, transition t is enabled in $(p, u_1, \ldots, u_k)$ if $\delta(t) = (p, Op, q)$ for some $q \in Q$ and some $Op \in OP_k$, and for all $i \in \{1, \ldots, k\}$, if $Op(i) = nop$ then $u_i = u'_i$, if $Op(i) = \epsilon?$ then $u_i = u'_i = \epsilon$, if $Op(i) = !a$ then $u'_i = u_i a$ and if $Op(i) = ?a$ then $u_i = a u'_i$, in which case t can occur, reaching state $(q, u'_1, \ldots, u'_k)$.

The semantics of LCS is given as the lossy version of the previous semantics, when considering the canonic order in $Q \times (A^*)^k$ for which LCS are WSTS.

If Σ_p is defined by $\Sigma_p = \{\alpha_1, ..., \alpha_p\}$ where α_i's are constant symbols, we define $LCS(k, p)$ as the subclass of LCS with k channels and set of messages Σ_p. We have:

Theorem 4. $LCS(k, p) \prec LCS(k + 1, p) \prec LCS(1, p + 1)$

Proof. $LCS(k, p) \preceq LCS(k+1, p)$ clearly holds. The proof that $LCS(k+1, p) \preceq LCS(1, p + 1)$ is based on the well-known fact that one can simulate the $k + 1$ channels by inserting a new symbol k times as delimiters. A proof is available in Appendix B. For the strictness, we remark again that the WSTS introduced in the proof of Prop. 2 is actually a LCS, that is, given a state space $X = Q \times (\Sigma_p^*)^k$, we can find $\mathcal{S}$ in $LCS(k, p)$ and a witness L of X such that $\mathcal{S}$ recognizes L. This implies that $LCS(k, p)$ is self-witnessing. For all k and p, $ot(Q \times (\Sigma_p^*)^k) = \omega^{\omega^{p-1} * k} * |Q|$. This implies that $(\Sigma_p^*)^{k+1} \not\sqsubseteq_{refl} Q \times (\Sigma_p^*)^k$ and $\Sigma_{p+1}^* \not\sqsubseteq_{refl} Q \times (\Sigma_p^*)^k$ for all Q. To conclude we only need to apply proposition 4.

Moreover, in [2] the authors prove that $AWN \prec LCS$. We can easily get back this result:

Proposition 8. $LCS(1, 2) \not\preceq AWN$.

Proof. As in the previous result, we remark that $LCS(1, 2)$ and AWN are self-witnessing. Thus, we only need to apply Prop. 4, considering that for any $k > 0$, $\Sigma_2^* \not\sqsubseteq_{refl} \mathbb{N}^k$ (Cor. 1).

This result is tight: $LCS(0, p) \simeq FA$ (Finite Automata), $LCS(k, 1) \simeq VASS_k$.

5 Petri Nets extensions with data

Many extensions of Petri nets with data have been defined in the literature to gain expressive power for better modeling capabilities. Data Nets (DN) [12] are a monotonic extension of Petri nets in which tokens are taken from a linearly

[3] This definition is a slight variation of the usal one in order to uniformise presentation of VASS and LCS without effect on their expressive power.

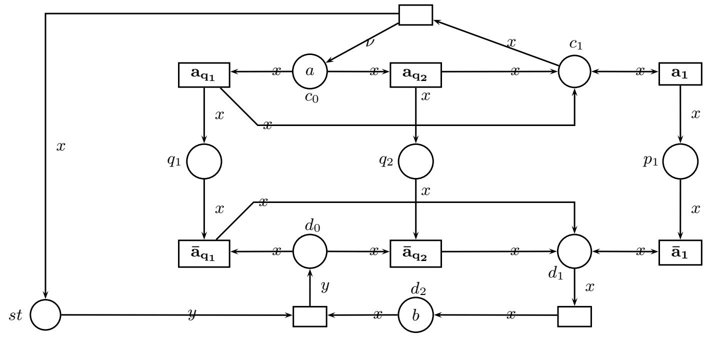

Fig. 1. Net in ν-PN_1 recognizing a witness of $(Q \times \mathbb{N})^{\oplus}$ with $|Q| = 2$

ordered and dense domain, and transitions can perform whole place operations like transfers, resets or broadcasts. A similar model, in which tokens can only be compared with equality, is that of ν-Petri Nets (ν-PN) [13]. The relative expressive power of DN and ν-PN has been an open problem since [14]. In this section we prove that ν-$PN \prec DN$. We work with the subclass of DN without whole place operations, called *Petri Data Net* (PDN), since $DN \simeq PDN$ [2].

Now we briefly define ν-PN. The definition of PDN is in Appendix C. We consider an infinite set Id of names, a set Var of variables and a subset of special variables $\Upsilon \subset Var$ for fresh name creation. A ν-PN is a tuple $N = (P, T, F, \Sigma, \lambda)$, where P and T are finite disjoint sets, $F : (P \times T) \cup (T \times P) \to Var^{\oplus}$, Σ is the finite labelling alphabet, and $\lambda : T \to (\Sigma \cup \{\epsilon\})$ labels transitions.

A *marking* is a mapping $M : P \to Id^{\oplus}$. A *mode* is an injection $\sigma : Var(t) \to Id$. A transition t can be fired with mode σ for a marking M if for all $p \in P$, $\sigma(F(p,t)) \subseteq M(p)$ and for every $\nu \in \Upsilon$, $\sigma(\nu) \notin M(p)$ for all p. In that case we have $M \xrightarrow{\lambda(t)} M'$, where $M'(p) = (M(p) - \sigma(F(p,t))) + \sigma(F(t,p))$ for all $p \in P$.

Markings can be identified up to renaming of names. Thus, markings of a ν-PN with k places can be represented as elements in $(\mathbb{N}^k)^{\oplus}$, each tuple representing the occurrences in each place of one name [15]. E.g., if $P = \{p_1, p_2\}$ and M is such that $M(p_1) = \{|a, a, b|\}$ and $M(p_2) = \{|b|\}$, then we can represent M as $\{|(2, 0), (1, 1)|\}$.

The i-th place of a ν-PN is *bounded* if every tuple $(n_1, ..., n_k)$ in every reachable marking satisfies $n_i \leq b$, for some $b \geq 0$. Therefore, a bounded place may contain arbitrarily many names, provided each of them appears a bounded number of times.

Let us denote by ν-PN_k the class of ν-PN with k unbounded places. If a net in ν-PN_k has m places bounded by some $b \geq 0$, then we can use as state space $(Q \times \mathbb{N}^k)^{\oplus}$ with $Q = \{0, ..., b\}^m$ (finite and non-empty). Thus, the state space of nets in ν-PN_k is in $\mathbf{X}_k^{\oplus} = \{(Q \times \mathbb{N}^k)^{\oplus} \mid Q \text{ finite}\}$. Analogously, the class PDN_k of PDN with k unbounded places has $\mathbf{X}_k^{*} = \{(Q \times \mathbb{N}^k)^{*} \mid Q \text{ finite}\}$ as set of state spaces. Moreover, we take $\mathbf{X}^{\oplus} = \{(\mathbb{N}^k)^{\oplus} \mid k > 0\}$ and $\mathbf{X}^{*} = \{(\mathbb{N}^k)^{*} \mid k > 0\}$.

Proposition 9. *For every $k \geq 0$, ν-PN_k and PDN_k are self-witnessing.*

Proof. The proof for PDN_k is in Appendix C. Let us see it for ν-PN_k. Let $(Q \times \mathbb{N}^k)^{\oplus} \in \mathbf{X}_k^{\oplus}$. We consider an alphabet $\Sigma = \{a_q \mid q \in Q\} \cup \{a_1, ..., a_k\}$ and we define $\gamma : \Sigma^* \to (Q \times \mathbb{N}^k)^\oplus$ by

$$\gamma(a_{q_1} a_1^{n_1^1} ... a_k^{n_1^k} .... a_{q_l} a_1^{n_l^1} ... a_k^{n_l^k}) = \{\!|(q_1, n_1^1, ..., n_1^k), ..., (q_l, n_l^1, ..., n_l^k)|\!\}$$

Let us build N in ν-PN_k such that $L(N) \cap dom(\gamma)\overline{dom(\gamma)} = L_\gamma$. Assume $Q = \{q_1, ..., q_r\}$. Fig. 1 shows the case with $k = 1$ and $r = 2$.

The only unbounded places of N are $p_1, ..., p_k$ (hence $N \in \nu$-PN_k). We consider $q_1, ..., q_r$ as places, a place st that stores all the names that have been used (once each name, hence bounded), and places $c_0, c_1, ..., c_k$ containing one name in mutual exclusion. When the name is in c_0 it is non-deterministically copied in some q (action labelled by a_q), and moved to c_1. For every, $1 \leq i \leq k$, when the name is in c_i it can be copied arbitrarily often to p_i (labelled by a_i). At any time, this name can be transferred to c_{i+1} when $i < k$ or to st for $i = k$ (labelled by ϵ). In the last case a fresh name is put in c_0 (thanks to $\nu \in \Upsilon$).

The second phase is analogous, with control places $d_0, d_1, ..., d_{k+1}$, marked in mutual exclusion with names taken from st. At any point, the name in d_{k+1} can be removed, and one name moved from st to d_0 (labelled by ϵ). That name must appear in some q. Thus, for each q we have a transition that removes the name from d_0 and q and puts it in d_1 (labelled by $\bar{a}_q$). For each $1 \leq i \leq k$, the name in d_i can be removed zero or more times from p_i (labelled by $\bar{a}_i$). At any point, the name is transferred from d_i to d_{i+1} (labelled by ϵ).

The initial and final marking is that with a name in c_0 and another name in d_{k+1} (and empty elsewhere). It holds that $L(N) \cap dom(\gamma)\overline{dom(\gamma)} = L_\gamma$, so we conclude.

Notice that since ν-PN_k and PDN_k are self-witnessing for every $k \geq 0$, so are ν-PN and PDN.

Proposition 10. *$\mathbf{X}_1^* \not\sqsubseteq_{refl} \mathbf{X}^\oplus$, $\mathbf{X}_{k+1}^\oplus \not\sqsubseteq_{refl} \mathbf{X}_k^\oplus$ and $\mathbf{X}_{k+1}^* \not\sqsubseteq_{refl} \mathbf{X}_k^*$ for all k.*

Proof. $\mathbf{X}_1^* \not\sqsubseteq_{refl} \mathbf{X}^\oplus$ holds because $ot(\mathbb{N}^*) = \omega^{\omega^\omega} \not\leq \omega^{\omega^k} = ot((\mathbb{N}^k)^\oplus)$, so that $\mathbb{N}^* \not\sqsubseteq_{refl} (\mathbb{N}^k)^\oplus$ for all k. The others are obtained similarly, considering that $ot((Q \times \mathbb{N}^k)^\oplus) = \omega^{\omega^k * |Q|}$ and $ot((Q \times \mathbb{N}^k)^*) = \omega^{\omega^{\omega^k * |Q|}}$.

Corollary 3. *ν-$PN \prec PDN$. Moreover, $PDN_1 \not\preceq \nu$-PN.*

Proof. ν-$PN \preceq PDN$ is from [14]. $PDN_1 \not\preceq \nu$-PN is a consequence of Prop. 4, considering that both classes are self-witnessing, and that $\mathbf{X}_1^* \not\sqsubseteq_{refl} \mathbf{X}^\oplus$.

We can even be more precise in the hierarchy of Petri Nets extensions.

Proposition 11. *For any $k \geq 0$, ν-$PN_k \prec \nu$-PN_{k+1} and $PDN_k \prec PDN_{k+1}$.*

Proof. Clearly ν-$PN_k \preceq \nu$-PN_{k+1} and $PDN_k \preceq PDN_{k+1}$ for any $k \geq 0$. For the converses, again we can apply Prop. 4, considering that all the classes considered are self-witnessing and that $\mathbf{X}_{k+1}^\oplus \not\sqsubseteq_{refl} \mathbf{X}_k^\oplus$ and $\mathbf{X}_{k+1}^* \not\sqsubseteq_{refl} \mathbf{X}_k^*$ hold.

Finally, we can strengthen the result $AWN \prec \nu$-PN proved in [14] in a very straightforward way.

Proposition 12. $\nu\text{-}PN_1 \not\preceq AWN$

Proof. Both AWN and $\nu\text{-}PN_1$ are self-witnessing, and $\mathbf{X}_1^{\oplus} \not\sqsubseteq_{refl} \{\mathbb{N}^k \mid k > 0\}$ because $\mathbb{N}^{\oplus} \not\sqsubseteq_{refl} \mathbb{N}^k$ for all k (indeed, $ot(\mathbb{N}^{\oplus}) = \omega^{\omega} \not\leq \omega^k = ot(\mathbb{N}^k)$). By Prop. 4 we conclude.

Again, the previous result is tight. Indeed, a $\nu\text{-}PN$ with no unbounded places can be simulated by a Petri net, so that $\nu\text{-}PN_0 \simeq VASS$.

6 Conclusion and Perspectives

To show a strict hierarchy of WSTS classes, we have proposed a generic method based on two principles: the ability of WSTS to recognize some specific witness languages linked to their state space, and the use of order theory to show the absence of order reflections from one wpo to another. This allowed us to unify some existing results, while also solving open problems. We summarize the current picture on expressiveness of WSTS below w.r.t number of resources and type of resources. On the other hand, showing equivalence between WSTS classes is a problem deeply linked to the semantics of the models, and hence that remains to be solved on a case-by-case basis.

Quantitative results. (All results are new.)

For every $k \in \mathbb{N}$ $VASS_k \prec VASS_{k+1} \not\preceq AWN_k$

For every $k, p \in \mathbb{N}$ $LCS(k,p) \prec LCS(k+1,p) \prec LCS(1,p+1)$

For every $k \in \mathbb{N}$ $\nu\text{-}PN_k \prec \nu\text{-}PN_{k+1}$ and $PDN_k \prec PDN_{k+1}$

Qualitative results. (New results are $\nu\text{-}PN \prec DN$ and $PDN \simeq TdPN$)

$VASS \prec \mathcal{M} \prec DN \simeq PDN \simeq TdPN$

where $\mathcal{M}$ is either $\nu\text{-}PN$ or LCS

TdPN [3] are Timed Petri nets and we have proved the related result in a companion report [5].

An interesting case that remains open is the relative expressiveness of LCS and $\nu\text{-}PN$. Their state space are quite distinct but their order type are the same for some values of their parameters. We conjecture that there is no reflection from one to the other, but such a proof would require more than order type analysis.

As all the models that we have studied in this paper use a state space whose order type is bounded by ϵ_0, it is tempting to look at WSTS that would use a greater state space. It is known that the Kruskal ordering has an order type greater than ϵ_0 [16], even for unlabelled binary trees. However, studies of WSTS based on trees have been quite scarce [11]. We believe some interesting problems might lie in this direction.

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A Complements to section 3

We introduce first a few additional notations that we need for the proof of this section.[4]

A.1 Preliminaries

Let A be a well-ordered set. $X \subseteq A$ is a directed subset of A if $\forall x, y \in X,\ \exists z \in X,\ x \leq z \wedge y \leq z$. A downward closed directed subset of A is called an irreductible ideal [5] of A. We denote by $Idl(A)$ the set of irreductible ideals of A.

Proposition 13. *Let A be a well-ordered set. Then any downward closed subset of A is a finite union of irreductible ideals.*

Note that $\nu_I : A \to Idl(A)$ given by $\nu_I(x) = \downarrow x$ is an order-embedding. Because of this, we will identify x with $\downarrow x$.

A.2 Representations of $\boldsymbol{Idl}(\mathbb{N}^k)$ and $\boldsymbol{Idl}(A^\oplus)$

Proposition 14. *Let $I \in Idl(\mathbb{N}^k)$. I can be written as $(x_1, ..., x_k)$ with $x_i \in \mathbb{N} \cup \{\omega\}$, and:*

$$(y_1, ..., y_k) \in (x_1, ..., x_k) \iff (\forall i,\ x_i \in \mathbb{N} \implies y_i \leq x_i)$$

For example, $(\omega, 4)$ denotes the subset of $\mathbb{N}^2$ whose elements are those with 4 or lower as their second coordinate. This can be seen as an extension of the classic ordinal representation, where $\omega = \mathbb{N}$.

Proposition 15. *Let $I \in Idl(A^\oplus)$. I can be written as $\{I_1^\omega, \cdots, I_p^\omega, J_1, \cdots, J_q\}$ where $I_1, ..., I_p, J_1, ..., J_q$ are irreductible ideals of A, and with*

$$x \in [\![\{I_1^\omega, \cdots, I_p^\omega, J_1, \cdots, J_q\}]\!]$$
$$\iff$$
$$\begin{cases} x = x_1 \cup ... \cup x_p \cup y_1 \cup ... \cup y_q \\ \forall 1 \leq k \leq p,\ a \in x_k \implies a \in I_k \\ \forall 1 \leq k \leq q,\ y_k = \emptyset \vee (y_k = \{a\} \wedge a \in J_k) \end{cases}$$

For example $\{1^\omega, 3\}$ describes the subset of $\mathbb{N}^\oplus$ whose elements are those that contain any number of 0 or 1, and at most one element equal to 2 or 3. Note that an irreductible ideal has more than one possible representation. We have for example $\{2^\omega, 1\} = \{2^\omega\}$.

[4] Although the notations vary slightly from "Forward Analysis for WSTS : Part I : Completions" by A. Finkel and J. Goubault-Larrecq (STACS '09), sections A.1 and A.2 are straight rewriting of results from this paper.

[5] Some authors have been using the term 'ideal' as a shortcut for either a downward closed subset, or for a directed one. To avoid any confusion, we will only speak of irreductible ideals and of downward closed subsets.

A.3 Additional Proofs

Proof of theorem 1 *Let X and Y be two wpo. We have:*

$$X \sqsubseteq_{refl} Y \implies WSTS_X \preceq WSTS_Y$$

Proof. Let $L = L(\mathcal{S}, x_0, x_f)$ for some WSTS $\mathcal{S}$ with state space X with initial and final states x_0 and x_f, respectively. We can assume that $\mathcal{S}$ is a lossy WSTS.

Let φ be a reflection from X to Y. Since φ is an injection, we can consider the following labelled transition system $\mathcal{S}^\varphi$, of states $\varphi(X) \subseteq Y$, with initial and final states $\varphi(x_0)$ and $\varphi(x_f)$, respectively, and whose transitions are defined by:

$$\varphi(x) \xrightarrow{u}_{\mathcal{S}^\varphi} \varphi(x') \iff x \xrightarrow{u}_{\mathcal{S}} x'$$

It holds that $\mathcal{S}^\varphi \in WSTS_Y$. Indeed, if we take $\varphi(x_1)$, $\varphi(x_1')$ and $\varphi(x_2)$ such that $\varphi(x_1) \xrightarrow{u} \varphi(x_1')$ and $\varphi(x_2) \geq \varphi(x_1)$, then we have by definition of $\mathcal{S}^\varphi$, and because φ is a reflection, that $x_1 \xrightarrow{u} x_1'$ and $x_2 \geq x_1$, which means, by well-structure of $\mathcal{S}$, that there exists $x_2' \geq x_1'$ such that $x_2 \xrightarrow{u} x_2'$. By the lossiness property of $\mathcal{S}$, we have $x_2 \xrightarrow{u} x_1'$, and thus $\varphi(x_2) \xrightarrow{u} \varphi(x_1')$. Moreover, $\mathcal{S}$ and $\mathcal{S}^\varphi$ clearly recognize the same language, so that $L = L(\mathcal{S}^\varphi, \varphi(x_0), \varphi(x_f))$ with $\mathcal{S}^\varphi \in WSTS_Y$, which concludes our proof.

Proposition 16. $\mathbb{N}^2 \sqsubseteq \mathbb{N}^\oplus$.

Proof. $\varphi : \mathbb{N}^2 \to \mathbb{N}^\oplus$ given by $\varphi(a, b) = \{\!|a + 2, 1^b|\!\}$ is an order-embedding.

Proposition 17. $\mathbb{N}^3 \not\sqsubseteq \mathbb{N}^\oplus$

Proof. Assume φ is an order-embedding from $\mathbb{N}^3$ to $\mathbb{N}^\oplus$.

We consider the following sets:

- $A_x = \{(n, 0, 0) \mid n \in \mathbb{N}\}$
- $A_y = \{(0, n, 0) \mid n \in \mathbb{N}\}$
- $A_z = \{(0, 0, n) \mid n \in \mathbb{N}\}$

For any $\alpha \in \{x, y, z\}$, $\varphi(A_\alpha)$ is an infinite chain of $\mathbb{N}^\oplus$ with limit an element of $Idl(\mathbb{N}^\oplus)$. If this element is the entire set, for any element x of $\mathbb{N}^3$, we can find an element x' of A_α such that $\varphi(x) \leq \varphi(x')$, contradicting the order embedding.

Thus, let $\{\omega^{k_\alpha}, {k'_\alpha}^\omega\} \cup B_\alpha$ be this element.

We remark that for any three pairs of integers, we can choose one of these pairs that is less or equal than the lub of the two others.

This means, that we can find α, β and γ, such that :

$$(k_\alpha, k'_\alpha) \leq (max\{k_\beta, k_\gamma\}, max\{k'_\beta, k'_\gamma\})$$

Without loss of generality, we will assume $\alpha = x$, $\beta = y$ and $\gamma = z$. Then, we define $A_{y,z}[a] = \{(a, n, n) | n \in \mathbb{N}\}$.

In the same way as before, we have the image of $A_{y,z}[a]$ an infinite chain of $\mathbb{N}^{\oplus}$, with limit $\{\omega^{k_{y,z}[a]}, (k'_{y,z}[a])^{\omega}\} \cup B_{y,z}[a]$. Because φ is an order embedding, for any $a \in \mathbb{N}$, this limit is greater than both $\{\omega^{k_y}, {k'_y}^{\omega}\} \cup B_y$ and $\{\omega^{k_z}, {k'_z}^{\omega}\} \cup B_z$, implying that :

$$\forall a \in \mathbb{N}, \quad k_x \leq k_{y,z}[a] \quad \text{and} \quad k'_x \leq k'_{y,z}[a]$$

As we have $\varphi(n,0,0) \rightarrow \omega^{k_x}.{k'_x}^{\omega}.B_x$, we can find an a_0 such that $\varphi(a_0,0,0) = \{p_1, \cdots, p_{k_x}, q_1, \cdots, q_r\} \cup B_x$ with :

- $r \in \mathbb{N}$
- $\forall 1 \leq i \leq k_x,\ p_i \geq max(k'_x, M)$, where M is the greatest value in B_x
- $\forall 1 \leq i \leq r,\ q_i \leq k'_x$

We define $P = \{p_1, \cdots, p_{k_x}\}$ and $Q = \{q_1, \cdots, q_r\}$. We have :

$$P \cup Q \cup B_x \leq \{\omega^{k_{y,z}[a_0]}, k'_{y,z}[a_0]^{\omega}\} \cup B_{y,z}[a_0]$$

Elements of P are bigger than all elements in Q and B_0, thus :

$$Q \cup B_x \leq \{\omega^{k_{y,z}[a_0]-k_x}, k'_{y,z}[a_0]^{\omega}\} \cup B_{y,z}[a_0]$$

Because $k'_x \leq k'_{y,z}[a_0]$, we have :

$$\begin{aligned} & \{{k'_x}^{\omega}\} \cup B_x \leq \{\omega^{k_{y,z}[a_0]-k_x}, k'_{y,z}[a_0]^{\omega}\} \cup B_{y,z}[a_0] \\ \Rightarrow \quad & \{\omega^{k_x}, {k'_x}^{\omega}\} \cup B_x \leq \{\omega^{k_{y,z}[a_0]}, k'_{y,z}[a_0]^{\omega}\} \cup B_{y,z}[a_0] \end{aligned}$$

and because that means that each image of an element of A_x can be compared to an element of $A_{y,z}[a_0]$, we get a contradiction that concludes the demonstration.

Proposition 18. *For any k, there is an order reflection from $\mathbb{N}^k$ to $\mathbb{N}^{\oplus}$*

Proof. Let us take a fixed $k \in \mathbb{N}$. There is a finite number of possible relative orders of $x_1, ..., x_k$. Let N_k be this number, and let o_k be a mapping that associates with each tuple $(x_1, ..., x_k)$ a number between 0 and $N_k - 1$ such that $o_k(x_1, ..., x_k) = o_k(x'_1, ..., x'_k)$ means that $x_1, ..., x_k$ and $x'_1, ..., x'_k$ are in the same relative order.

We define $ac\ : \mathbb{N} \rightarrow \mathbb{N}^{\oplus}$ by $ac(n) = \{\!|2N_k - (n+1), n|\!\}$. Note that $ac(m)$ and $ac(n)$ are incomparable with respect to the multiset order if m and n are different numbers between 0 and $N_k - 1$.

Now we define φ by :

$$\varphi(x_1, ...x_k) = \{\!|(2N_k + x_1), (2N_k + x_2), \cdots, (2N_k + x_k)|\!\} + ac(o_k(x_1, ...x_k))$$

We claim this is an order reflection.

Indeed, let us take $X = (x_1, ..., x_k)$ and $X' = (x'_1, ..., x'_k)$ and assume that we have $\varphi(X) \leq_{\mathbb{N}^{\oplus}} \varphi(X')$. Then, there is a bijective mapping $\sigma\ :$

$$\sigma : \varphi(X) \to \varphi(X')$$

with :

$$\varphi(X) = \{\!|2N_k + x_1, ..., 2N_k + x_k, 2N_k - (o_k(X)+1), o_k(X)|\!\}$$
$$\varphi(X') = \{\!|2N_k + x'_1, ..., 2N_k + x'_k, 2N_k - (o_k(X')+1), o_k(X')|\!\}$$
$$\forall x \in \varphi(X).\ x \leq \sigma(x)$$

The cardinality of $\varphi(X)$ and $\varphi(X')$ are the same, and the elements of the form $2N_k + x_i$ can only be mapped to one of their counterpart, so :

$$\sigma(2N_k - (o_k(X)+1)) = 2N_k - (o_k(X')+1)$$
$$\sigma(o_k(X)) = o_k(X')$$

This means that $o_k(X) = o_k(X')$. The components of X and X' are thus in the same relative order. Without loss of generality, we will assume this order is $x_1 \leq x_2 \leq ... \leq x_k$. Let us assume that there exists i such that $x_j \leq x'_j$ for all $j > i$ and $x_i > x'_i$. Then, we have $x_i \leq x'_m$ for some m.

Two cases may occur :

- $\underline{m > i}$: Then by cardinality, we have an element x_p in $\{x_{i+1}, ..., x_k\}$ that is mapped to an element $x'_{p'}$ with $p' \leq i$. Thus, we have $x_i \leq x_p \leq x'_{p'} \leq x'_i$, contradicting our hypothesis that $x'_i < x_i$.
- $\underline{m < i}$: Then, we have $x_i \leq x'_m \leq x'_i$, contradicting again our hypothesis.

Thus, we have $x_i \leq x'_i$ for all i, concluding our demonstration.

Proof of proposition 2 *Let X be a set produced by the grammar Γ. Then, there is a witness of X that is recognized by a WSTS of state space X.*

Proof. We have $X = Q \times \Sigma_1^* \times \cdots \times \Sigma_k^*$, ordered by its canonic order $\leq_X$ (which is the cartesian product of equality on Q and subword ordering on the alphabets Σ_i^* for all i). Without loss of generality, we will assume that the Σ_i's are disjoint. We also define $\Sigma_T = \bigcup_{1 \leq i \leq k} \Sigma_i$ and we choose arbitrarily a $q_0 \in Q$. Finally, we define $\Sigma_Q = \{a_q \mid q \in Q\}$, also disjoint from Σ_T.

We define a WSTS $\mathcal{S} = \langle X, \Sigma, \to, \leq_X \rangle$ by:

- $\Sigma = \Sigma_T \cup \Sigma_Q \cup \overline{\Sigma}_T \cup \overline{\Sigma}_Q$
- For $a \in \Sigma_T$, $(q, u_1, ..., u_k) \xrightarrow{a} (q', u'_1, ..., u'_k) \iff \begin{cases} q = q' & \\ u'_i = u_i a & \text{if } a \in \Sigma_i \\ u'_j = u_j & \text{otherwise} \end{cases}$
- For $\overline{a} \in \overline{\Sigma}_T$, $(q, u_1, ..., u_k) \xrightarrow{\overline{a}} (q', u'_1, ..., u'_k) \iff \begin{cases} q = q' & \\ u_i = a u'_i & \text{if } a \in \Sigma_i \\ u_j = u'_j & \text{otherwise} \end{cases}$
- For $a_{q_s} \in \Sigma_Q$, $(q, u_1, ..., u_k) \xrightarrow{a_{q_s}} (q', u'_1, ..., u'_k) \iff \begin{cases} q = q_0 \\ q' = q_s \\ u'_i = u_i \end{cases}$

– For $\overline{a_{q_s}} \in \Sigma_Q$, $(q, u_1, ..., u_k) \xrightarrow{\overline{a_{q_s}}} (q', u'_1, ..., u'_k) \iff \begin{cases} q = q_s \\ q' = q_0 \\ u'_i = u_i \end{cases}$

– $s \xrightarrow{\epsilon} s' \iff s' \leq s$

We define $\gamma(x) = (q, u_1, ..., u_k)$ iff $x \in a_q \| u_1 \| \cdots \| u_k$, where $\|$ denotes the shuffling operation (i.e. $z \in u \| v \iff z = u_1 v_1 u_2 \cdots u_p v_p$ with $u = u_1 u_2 \cdots u_p$ and $v = v_1 v_2 \cdots v_p$, with $u_i, v_i \in \Sigma^*$). γ is a $(\Sigma_T \cup \Sigma_Q)$-representation of X.

We define $L = L(\mathcal{S}, (q_0, \epsilon, ..., \epsilon), (q_0, \epsilon, ..., \epsilon))$ and we have:

$$L \cap dom(\gamma)\overline{dom(\gamma)} = \{u\overline{v} \mid u, v \in dom(\gamma) \text{ and } \gamma(v) \leq \gamma(u)\}$$

This concludes the demonstration.

Proof of proposition 5

$$ot(X) + 1 = sup\ \{\alpha \mid \exists f : \alpha \to Down(X),\ f \textit{ strictly increasing}\}$$

Proof.
We first prove that $ot(X)+1 \leq sup\ \{\alpha \mid \exists f : \alpha \to Down(X),\ f \text{ strictly increasing}\}$
Let $\leq'$ be a linearization of $\leq$ of order type $ot(X)$. Let φ be an isomorphism from $ot(X)$ to $(X, \leq')$. We define $f : ot(X) + 1 \to Down(X)$ by:

$$\begin{aligned} f(\beta) &= \{x \in X \mid x <' \varphi(\beta)\} \qquad \text{for } \beta < ot(X) \\ f(ot(X)) &= X \end{aligned}$$

f is strictly increasing, which means that:
$ot(X) + 1 \in \{\alpha \mid \exists f : \alpha \to Down(X),\ f \text{ strictly increasing}\}$ and concludes the first part of the proof.

We then prove that $ot(X)+1 \geq sup\ \{\alpha \mid \exists f : \alpha \to Down(X),\ f \text{ strictly increasing}\}$
Let α be an ordinal and f be a strictly increasing mapping from α to $Down(X)$. We define the quasi-order $\leq_f$ on X by:

$$x \leq_f y \text{ iff } \forall \beta < \alpha,\ y \in f(\beta) \implies x \in f(\beta)$$

$\leq_f$ is clearly reflexive and transitive. Let $\leq_{tie}$ be a linearization of $\leq_X$. We define the order $\leq'_f$ by:

$$x \leq'_f y \iff \begin{cases} x \leq_f y \wedge y \not\leq_f x \quad \text{or,} \\ x \leq_f y \wedge y \leq_f x \wedge x \leq_{tie} y \end{cases}$$

$\leq'_f$ is clearly reflexive and antisymmetric. Let's show transitivity. Assume that $x \leq'_f y$ and $y \leq'_f z$. If they are all three in the same equivalent class (resp. in different equivalent classes) of $\equiv_{\leq_f}$, $x \leq'_f z$ comes from transitivity of $\leq_{tie}$ (resp. $\leq_f$). If x and y are $\leq_f$-equivalent, and $y <_f z$ we immediately get $x <'_f z$. The last case is similar.

Let us prove that $\leq'_f$ is a linear order. Pick any x and y. If they are equivalent w.r.t. $\leq_f$, we get the result by linearity of $\leq_{tie}$. So assume by symmetry that there exists β, $x \in f(\beta)$ and $y \notin f(\beta)$. Then for any β' such that $y \in f(\beta')$, $\beta < \beta'$ since f is strictly increasing. Thus $x \in f(\beta')$. Since β' is arbitrary, this shows that $x \leq'_f y$.

Let us prove that $\leq'_f$ is a linearization of $\leq_X$. Pick any $x \leq_X y$ (and thus $x \leq_{tie} y$). Because for all β, $f(\beta)$ is downward closed, we have $x \leq_f y$, which leads to $x \leq'_f y$.

Choose some $x_{max} \notin X$, and $X' = X \cup \{x_{max}\}$. We extend $\leq'_f$ on X' by $x \leq'_f x_{max}$ for all $x \in X$. We define $\varphi : \alpha \to (X', \leq'_f)$ by:

$$\varphi(\beta) = \min_{\leq'_f}\{x \in X' \mid x \notin f(\beta)\}$$

The min is defined because X' is well-ordered and at least $x_{max} \notin f(\beta)$ for any β. Because f is strictly increasing, φ is also strictly increasing.

Let us show that φ is an order embedding. Assume $\beta < \beta'$. Then there exists y such that $y \in f(\beta')$ and $y \notin f(\beta)$. This means $\varphi(\beta) \leq'_f y$. As $y \in f(\beta')$ and $f(\beta')$ is downward closed, $\varphi(\beta) \in f(\beta')$, which implies $\varphi(\beta) < \varphi(\beta')$.

We have an order embedding from α to $(X', \leq'_f)$ which means $\alpha \leq ot(X') = ot(X) + 1$.

Lemma 1. *Let X and Y be two wpos and φ a reflection from X to Y. Let $A \subsetneq X$ with $A = \downarrow A$. Then $\downarrow\varphi(A) \subsetneq Y$*

Proof. Let us assume that $\downarrow\varphi(A) = Y$. Let us take $x \in X$, $x \notin A$. Since $\varphi(x) \in Y$ and $\downarrow\varphi(A) = Y$, there is $x' \in A$ such that $\varphi(x) \leq \varphi(x')$. Since φ is a reflection we have $x \leq x'$ and since A is downward closed $x \in A$, hence the contradiction.

Proof of proposition 6 *Let X and Y be two wpos. $X \sqsubseteq_{refl} Y \implies ot(X) \leq ot(Y)$.*

Proof. Let $\varphi : X \to Y$ be a reflection and let us consider an ordinal α and a mapping $f : \alpha \to Down(X)$, strictly increasing. We define $g : \alpha \to Down(Y)$ by $g(\beta) = \downarrow\varphi(f(\beta))$. By Lemma 1, g is strictly increasing. By the characterization of order type in Prop. 5, we have $ot(X) \leq ot(Y)$.

B Complements to section 4

Proposition 19. *Let $\mathcal{S}$ be a Lossy Channel System in $LCS(k,p)$. There exists a Lossy Channel System $\mathcal{S}'$ in $LCS(1,p+1)$ such that $L(\mathcal{S}) = L(\mathcal{S}')$.*

Proof. We order the k channels of $\mathcal{S}$, $C_1, \ldots, C_k$. We define recursively $C_{k+i} = C_i$. We keep a notion of "active channel" through the control states. A state of $\mathcal{S}'$ is $(q, i, u_i a_{p+1} u_{i+1} a_{p+1} \ldots a_{p+1} u_{i+k-1})$ where q is the original control state of $\mathcal{S}$, $1 \leq i \leq k$ is the current channel and u_i is the contents of channel C_i. Reading a character in C_i requires i to be the active channel, writing a character in C_i requires C_{i+1} to be the active channel.

The system can change the active channel from C_i to C_j $(j > i)$ at any time by iterating $j - i$ times the following sequence of ϵ-transitions:

– Write a_{p+1}
– Read a word in $\{a_1, \ldots, a_p\}^*$ and copy it to the end of the channel.
– Read a_{p+1}

As long as exactly $k-1$ separators a_{p+1} stay in the channel, the described system simulate $\mathcal{S}$. However, one can loose these separators. To remove spurious traces, we add a final checking procedure, starting from the final states of $\mathcal{S}$, that reads $k-1$ symbols a_{p+1} and, if successful, puts the system in its real final state.

C Complements to section 5

C.1 Definition of Petri Data Nets

We denote by $\mathbf{0}$ the null vector in any $\mathbb{N}^k$ and for a word $w = x_1 \cdots x_n$ we write $|w| = n$ and $w(i) = x_i$. A *PDN* is a tuple $N = (P, T, F, H, \lambda)$ where P is a finite set of places, T is a finite set of transitions, $\lambda : T \to (\Sigma \cup \{\epsilon\})$ and for each $t \in T$, $F_t, H_t \in (\mathbb{N}^{|P|})^*$ with $|F_t| = |H_t|$. A marking s of N is a finite sequence of vectors in $\mathbb{N}^{|P|} \setminus \mathbf{0}$. A marking s is the $\mathbf{0}$-contraction of a sequence $s' \in (\mathbb{N}^{|P|})^*$ if it can be obtained by removing all the occurrences of $\mathbf{0}$ from s'. We write $s_1 \overset{\lambda(t)}{\to} s_2$ for $t \in T$ with $|F_t| = |H_t| = n$ if:

– s_1 is the $\mathbf{0}$-contraction of $u_0 x_1 u_1 \cdots u_{n-1} x_n u_n$ with $u_i \in (\mathbb{N}^{|P|} \setminus \mathbf{0})^*$ and $x_i \in \mathbb{N}^{|P|}$,
– $x_i \geq F_t(i)$ and $y_i = (x_i - F_t(i)) + H_t(i)$ for $i \in \{1, \ldots, n\}$,
– s_2 is the $\mathbf{0}$-contraction of $u_0 y_1 u_1 \cdots u_{n-1} y_n u_n$

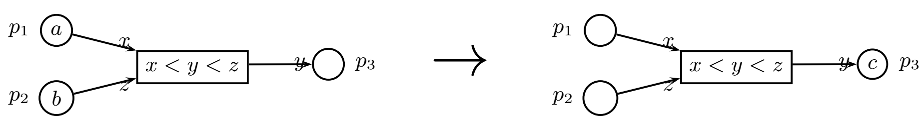

Fig. 2. Firing of a Petri Data Net transition (assuming $a < c < b$)

A Petri Data Net can be graphically depicted similarly as ν-*PN*, by considering that tokens carry data taken from a linearly ordered and dense domain, and arcs are labelled with variables (not in Υ) that are totally ordered. If $x_1 < \ldots < x_n$ are all the variables adjacent to a transition t, then $F_t(i)$ specifies the tokens that must be taken from each place carrying the datum to which x_i is instantiated (and analogously for H_t). For instance, Fig. 2 depicts a *PDN* with a single transition t given by $F_t = (1,0,0)(0,0,0)(0,1,0)$ and $H_t = (0,0,0)(0,0,1)(0,0,0)$.

Finally, let us remark that we can work with an "extension" of *PDN* in which variables adjacent to a transition are not necessarily totally ordered. Any such *PDN* can be simulated by an ordinary *PDN*. For instance, we can simulate a transition t in which two unrelated variables x and y appear by having a non-deterministic choice between two transitions t_1 and t_2, the former assuming $x < y$ and the latter assuming $y < x$. Analogously, we can have variables x and y so that $x \leq y$, that can be simulated again by having a non-deterministic choice between $x = y$ (that is, actually using the same variable) and $x < y$.

C.2 Additional proofs

Proof of proposition 9 For every $k \geq 0$, PDN_k is self-witnessing.

Proof. The case of PDN_k is analogous to that of ν-PN_k. Let $(Q \times \mathbb{N}^k)^* \in \mathbf{X}_k^*$. We define $\Sigma = \{a_q \mid q \in Q\} \cup \{a_1, ..., a_k\}$ and $\gamma : \Sigma^* \to (Q \times \mathbb{N}^k)^*$ by

$$\gamma(a_{q_1} a_1^{n_1^1} ... a_k^{n_1^k} .... a_{q_l} a_1^{n_l^1} ... a_k^{n_l^k}) = (q_1, n_1^1, ..., n_1^k) ... (q_l, n_l^1, ..., n_l^k)$$

The net N in PDN_k that we build is similar to the ν-PN we built in the case of ν-PN_k, except for two differences: On the one hand, whenever a fresh name was put in c_0, now we put a *greater* name (that is, we replace ν by a variable y such that $x < y$). On the other hand, whenever we took from *st* another name, now we take a greater name (that is, we assume $x < y$). Finally, the initial and final marking is that with one name in c_0 and a smaller name in d_{k+1}. Again, it holds that $L(N) \cap dom(\gamma)\overline{dom(\gamma)} = L_\gamma$, and we conclude.