

# Stochastic Petri Net

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- 1 Stochastic Petri Net
- 2 Markov Chain
- 3 Markovian Stochastic Petri Net
- 4 Generalized Markovian Stochastic Petri Net (GSPN)
- 5 Product-form Petri Nets

# Outline

## 1 Stochastic Petri Net

Markov Chain

Markovian Stochastic Petri Net

Generalized Markovian Stochastic Petri Net (GSPN)

Product-form Petri Nets

# Stochastic Petri Net versus Time Petri Net

- In TPN, the delays are *non deterministically* chosen.
- In Stochastic Petri Net (SPN), the delays are *randomly* chosen by sampling distributions associated with transitions.

... but these distributions are not sufficient to eliminate non determinism.

## Policies for a net

One needs to define:

- The *choice* policy.  
What is the next transition to fire?
- The *service* policy.  
What is the influence of the enabling degree of a transition on the process?
- The *memory* policy.  
What become the samplings of distributions that have not be used?

# Choice Policy

In the net, associate a distribution  $D_i$  and a weight  $w_i$  with every transition  $t_i$ .

Preselection w.r.t. a marking  $m$  and enabled transitions  $T_m$

- Normalize weights  $w_i$  of the enabled transitions:  $w'_i \equiv \frac{w_i}{\sum_{t_j \in T_m} w_j}$
- Sample the distribution defined by the  $w'_i$ 's.
- Let  $t_i$  be the selected transition, sample  $D_i$  giving the value  $d_i$ .

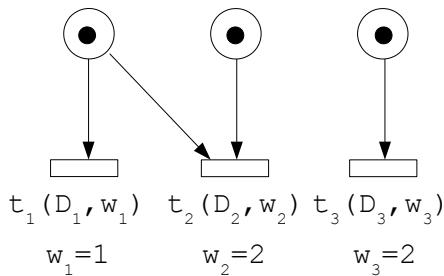
*versus*

**Race policy** with postselection w.r.t. a marking  $m$

- For every  $t_i \in T_m$ , sample  $D_i$  giving the value  $d_i$ .
- Let  $T'$  be the subset of  $T_m$  with the smallest delays.  
Normalize weights  $w_i$  of transitions of  $T'$ :  $w'_i \equiv \frac{w_i}{\sum_{t_j \in T'} w_j}$
- Sample the distribution defined by the  $w'_i$ 's yielding some  $t_i$ .

Priorities between transitions could be added to refine the selection.

# Choice Policy: Illustration



Preselection

Race Policy

Sample  $(1/5, 2/5, 2/5)$

Sample  $(D_1, D_2, D_3)$

Outcome  $t_1$

Outcome  $(\mathbf{3.2}, 6.5, \mathbf{3.2})$

Sample  $D_1$

Sample  $(1/3, -, 2/3)$

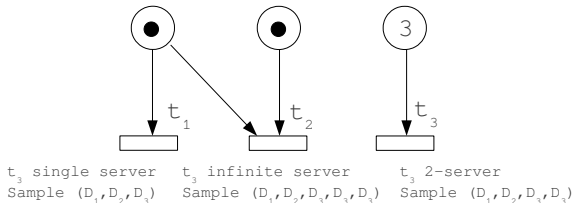
Outcome  $\mathbf{4.2}$

Outcome  $t_3$

# Server Policy

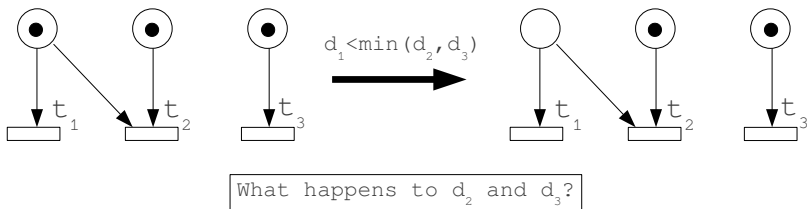
A transition  $t$  can be viewed as server for firings:

- A *single server*  $t$  allows a single instance of firings in  $m$  if  $m[t]$ .
- An *infinite server*  $t$  allows  $d$  instances of firings in  $m$  where  $d = \min\left(\left\lfloor \frac{m(p)}{Pre(p,t)} \right\rfloor \mid p \in \bullet t\right)$  is the *enabling degree*.
- A *multiple server*  $t$  with bound  $b$  allows  $\min(b, d)$  instances of firings in  $m$ .



This can be generalised by marking-dependent services.

# Memory Policy (1)

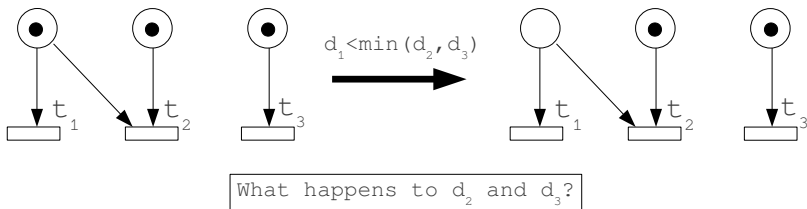


## Resampling Memory

Every sampling not used is forgotten.

This could correspond to a “crash” transition.

# Memory Policy (2)



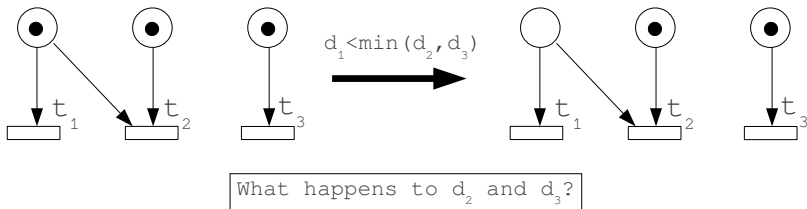
## Enabling Memory

- The samplings associated with still enabled transitions are kept and decremented ( $d'_3 = d_3 - d_1$ ).
- The samplings associated with disabled transitions are forgotten (like  $d_2$ ).

Disabling a transition could correspond to abort a service.



# Memory Policy (3)



## Age Memory

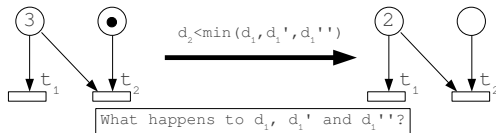
- All the samplings are kept and decremented ( $d'_3 = d_3 - d_1$  and  $d'_2 = d_2 - d_1$ ).
- The sampling associated with a disabled transition is frozen until the transition becomes again enabled (like  $d'_2$ ).

Disabling a transition could correspond to suspend a service.

# Memory Policy (4)

## Specification of memory policy

To be fully expressive, it should be defined w.r.t. any pair of transitions.



## Interaction between memory policy and service policy

Assume enabling memory for  $t_1$  when firing  $t_2$  and infinite server policy for  $t_1$ . Which sample should be forgotten?

- The last sample performed,
- The first sample performed,
- The greatest sample, etc.

**Warning:** This choice may have a critical impact on the complexity of analysis.

# Outline

## Stochastic Petri Net

### 2 Markov Chain

## Markovian Stochastic Petri Net

## Generalized Markovian Stochastic Petri Net (GSPN)

## Product-form Petri Nets

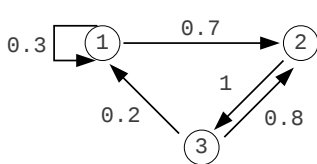
# Discrete Time Markov Chain (DTMC)

A DTMC is a stochastic process which fulfills:

- For all  $n$ ,  $T_n$  is the constant 1
- The process is *memoryless*

$$\begin{aligned} Pr(S_{n+1} = s_j \mid S_0 = s_{i_0}, \dots, S_{n-1} = s_{i_{n-1}}, S_n = s_i) \\ = Pr(S_{n+1} = s_j \mid S_n = s_i) \\ \equiv P[i, j] \end{aligned}$$

A DTMC is defined by  $S_0$  and  $P$



$$\mathbf{P} \begin{pmatrix} 0.3 & 0.7 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.2 & 0.8 & 0.0 \end{pmatrix}$$

# Analysis: the State Status

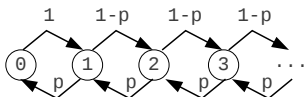
The *transient analysis* is easy and effective in the finite case:

$$\pi_n = \pi_0 \cdot P^n \text{ with } \pi_n \text{ the distribution of } S_n$$

The *steady-state analysis* ( $\exists \lim_{n \rightarrow \infty} \pi_n$ ) requires theoretical developments.

## Classification of states w.r.t. the asymptotic behaviour of the DTMC

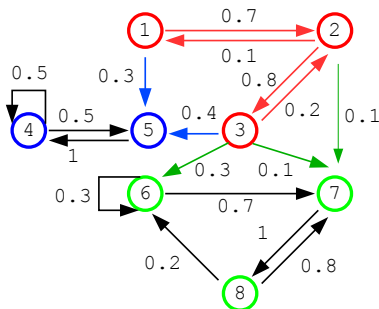
- A state is *transient* if the probability of a return after a visit is less than one. Hence the probability of its occurrence will go to zero. ( $p < 1/2$ )
- A state is *recurrent null* if the probability of a return after a visit is one but the mean time of this return is infinite. Hence the probability of its occurrence will go to zero. ( $p = 1/2$ )
- A state is *recurrent non null* if the probability of a return after a visit is one and the mean time of this return is finite. ( $p > 1/2$ )



# State Status in Finite DTMC

## In a finite DTMC

- The status of a state **only depends on the graph** associated with the chain.
- A state is transient iff it belongs to a non terminal *strongly connected component* (scc) of the graph.
- A state is recurrent non null iff it belongs to a terminal scc.



$$T = \{1, 2, 3\}$$

$$C_1 = \{4, 5\}$$

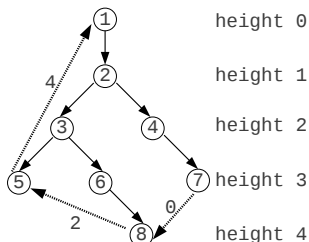
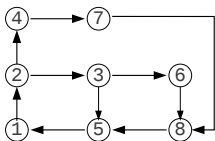
$$C_2 = \{6, 7, 8\}$$

# Analysis: Irreducibility and Periodicity

## Irreducibility and Periodicity

- A chain is *irreducible* if its graph is strongly connected.
- The *periodicity* of an irreducible chain is the greatest integer  $p$  such that: the set of states can be partitioned in  $p$  subsets  $\mathcal{S}_0, \dots, \mathcal{S}_{p-1}$  where every transition goes from  $\mathcal{S}_i$  to  $\mathcal{S}_{i+1 \% p}$  for some  $i$ .

### Computation of the periodicity



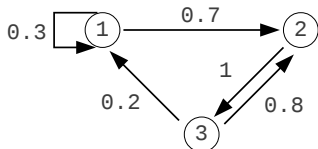
$$\text{periodicity} = \text{gcd}(0, 2, 4) = 2$$

# Analysis of a DTMC: a Particular Case

## A particular case

The chain is irreducible and *aperiodic* (i.e. its periodicity is 1)

- $\pi_\infty \equiv \lim_{n \rightarrow \infty} \pi_n$  exists and its value is independent from  $\pi_0$ .
- $\pi_\infty$  is the unique solution of  $X = X \cdot P \wedge X \cdot \mathbf{1} = 1$  where one can omit an arbitrary equation of the first system.



$$\pi_\infty = \left( 1/8 \quad 7/16 \quad 7/16 \right)$$

$$\pi_1 = 0.3\pi_1 + 0.2\pi_2 \quad \pi_2 = 0.7\pi_1 + 0.8\pi_3 \quad \pi_3 = \pi_2$$



# Analysis of a DTMC: the “General” Case

Almost general case: every terminal scc is aperiodic

- $\pi_\infty$  exists.
- $\pi_\infty = \sum_{s \in S} \pi_0(s) \sum_{i \in I} \text{preach}_i[s] \cdot \pi_\infty^i$  where:
  - 1  $S$  is the set of states,
  - 2  $\{\mathcal{C}_i\}_{i \in I}$  is the set of terminal scc,
  - 3  $\pi_\infty^i$  is the steady-state distribution of  $\mathcal{C}_i$ ,
  - 4 and  $\text{preach}_i[s]$  is the probability to reach  $\mathcal{C}_i$  starting from  $s$ .

Computation of the reachability probability for transient states

- Let  $T$  be the set of transient states  
(i.e. not belonging to a terminal scc)
- Let  $P_{T,T}$  be the submatrix of  $P$  restricted to transient states
- Let  $P_{T,i}$  be the submatrix of  $P$  transitions from  $T$  to  $\mathcal{C}_i$
- Then  $\text{preach}_i = (\sum_{n \in \mathbb{N}} (P_{T,T})^n) \cdot P_{T,i} \cdot \mathbf{1} = (Id - P_{T,T})^{-1} \cdot P_{T,i} \cdot \mathbf{1}$

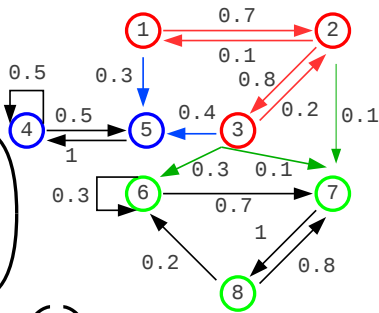
# Illustration: SCC and Matrices

$T = \{1, 2, 3\}, C_1 = \{4, 5\}, C_2 = \{6, 7, 8\}$

$$P_{T,T} = \begin{pmatrix} 0.0 & 0.7 & 0.0 \\ 0.1 & 0.0 & 0.8 \\ 0.0 & 0.2 & 0.0 \end{pmatrix}$$

$$P_{T,1} \cdot \mathbf{1} = \begin{pmatrix} 0.0 & 0.3 \\ 0.0 & 0.0 \\ 0.0 & 0.4 \end{pmatrix} \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.0 \\ 0.4 \end{pmatrix}$$

$$P_{T,2} \cdot \mathbf{1} = \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.1 & 0.0 \\ 0.3 & 0.1 & 0.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 0.0 \\ 0.1 \\ 0.4 \end{pmatrix}$$



# Continuous Time Markov Chain (CTMC)

A CTMC is a stochastic process which fulfills:

- Memoryless state change

$$\begin{aligned} Pr(S_{n+1} = s_j \mid S_0 = s_{i_0}, \dots, S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, \dots, T_n < \tau_n, S_n = s_i) \\ = Pr(S_{n+1} = s_j \mid S_n = s_i) \equiv P[i, j] \end{aligned}$$

- Memoryless transition delay

$$\begin{aligned} Pr(T_n < \tau \mid S_0 = s_{i_0}, \dots, S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, \dots, T_{n-1} < \tau_{n-1}, S_n = s_i) \\ = Pr(T_n < \tau \mid S_n = s_i) = 1 - e^{-\lambda_i \tau} \end{aligned}$$

## Notations and properties

- P defines an *embedded* DTMC (the chain of state changes)
- Let  $\pi(\tau)$  the distribution de  $X(\tau)$ , for  $\delta$  going to 0 it holds that:  
$$\pi(\tau + \delta)(s_i) \approx \pi(\tau)(s_i)(1 - \lambda_i \delta) + \sum_j \pi(\tau)(s_j) \lambda_j \delta P[j, i]$$

- Hence, let Q *the infinitesimal generator* defined by:

$$Q[i, j] \equiv \lambda_j P[i, j] \text{ for } j \neq i \text{ and } Q[i, i] \equiv - \sum_{j \neq i} Q[i, j]$$

Then:

$$\frac{d\pi}{d\tau} = \pi \cdot Q$$

# The exponential distribution

Let  $F$  be defined by:  $F(\tau) = 1 - e^{-\lambda\tau}$

Then  $F$  is the *exponential distribution* with rate  $\lambda > 0$ .

The exponential distribution is *memoryless*.

Let  $X$  be a random variable with a  $\lambda$ -exponential distribution.

$$\Pr(X > \tau' \mid X > \tau) = \frac{\Pr(X > \tau')}{\Pr(X > \tau)} = \frac{e^{-\lambda\tau'}}{e^{-\lambda\tau}} = e^{-\lambda(\tau' - \tau)} = \Pr(X > \tau' - \tau)$$

The minimum of exponential distributions is an exponential distribution.

Let  $Y$  be independent from  $X$  with  $\mu$ -exponential distribution.

$$\Pr(\min(X, Y) > \tau) = e^{-\lambda\tau} e^{-\mu\tau} = e^{-(\lambda + \mu)\tau}$$

The minimal variable is selected proportionally to its rate.

$$\Pr(X < Y) = \int_0^{\infty} \Pr(Y > \tau) F_X\{d\tau\} = \int_0^{\infty} e^{-\mu\tau} \lambda e^{-\lambda\tau} d\tau = \frac{\lambda}{\lambda + \mu}$$

# Convoluting the exponential distribution

The  $n^{\text{th}}$  convolution of a distribution  $F$  is defined by:

$$F^{n\star} \stackrel{\text{def}}{=} F \star \dots \star F \quad (n \text{ times})$$

Let  $f_n$  (resp.  $F_n$ ) be the density (resp. distribution) of the  $n^{\text{th}}$  convolution of the  $\lambda$ -exponential distribution. Then:

$$f_n(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} \quad \text{and} \quad F_n(x) = 1 - e^{-\lambda x} \sum_{0 \leq m < n} \frac{(\lambda x)^m}{m!}$$

## Sketch of proof

Recall that:  $f_1(x) = \lambda e^{-\lambda x}$ .

$$\begin{aligned} f_{n+1}(x) &= \int_0^x f_n(x-u) f_1(u) du = \int_0^x \lambda e^{-\lambda(x-u)} \frac{(\lambda(x-u))^{n-1}}{(n-1)!} \lambda e^{-\lambda u} du \\ &= \lambda e^{-\lambda x} \int_0^x \lambda \frac{(\lambda(x-u))^{n-1}}{(n-1)!} du = \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!} \end{aligned}$$

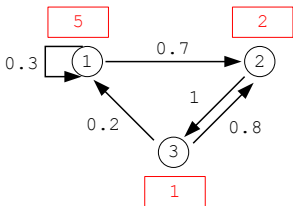
Deduce  $F_{n+1}$  by:

$$\begin{aligned} &\frac{d}{dx} \left( 1 - e^{-\lambda x} \sum_{0 \leq m \leq n} \frac{(\lambda x)^m}{m!} \right) = \\ &e^{-\lambda x} \left( \lambda \sum_{0 \leq m \leq n} \frac{(\lambda x)^m}{m!} - \sum_{0 \leq m \leq n-1} \lambda \frac{(\lambda x)^m}{m!} \right) = f_{n+1}(x) \end{aligned}$$

# CTMC: Illustration and Uniformization

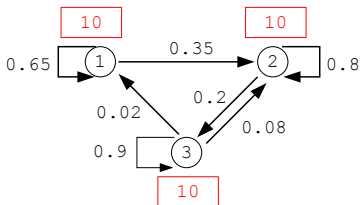
## A CTMC

$\lambda$

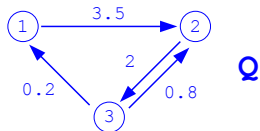


$P$

$\lambda'$



$P'$



*A uniform version of the CTMC (equivalent w.r.t. the states)*

# Analysis of a CTMC

## Transient Analysis

- Construction of a uniform version of the CTMC  $(\lambda, P)$  such that  $P[i, i] > 0$  for all  $i$ .
- Computation by case decomposition w.r.t. the number of transitions:

$$\pi(\tau) = \pi(0) \sum_{n \in \mathbb{N}} (e^{-\lambda\tau}) \frac{\tau^n}{n!} P^n$$

## Steady-state analysis

- The steady-state distribution of visits is given by the steady-state distribution of  $(\lambda, P)$  (by construction, the terminal scc are aperiodic) ...
- equal to the steady-state distribution since the sojourn times follow the same distribution.
- A particular case:  $P$  irreducible  
the steady-state distribution  $\pi$  is the unique solution of  $X \cdot Q = 0 \wedge X \cdot \mathbf{1} = 1$  where one can omit an arbitrary equation of the first system.

# Outline

Stochastic Petri Net

Markov Chain

3 Markovian Stochastic Petri Net

Generalized Markovian Stochastic Petri Net (GSPN)

Product-form Petri Nets



# Markovian Stochastic Petri Net

## Hypotheses

- The distribution of every transition  $t_i$  has a density function  $e^{-\lambda_i \tau}$  where the parameter  $\lambda_i$  is called *the rate* of the transition.
- **For simplicity reasons**, the server policy is single server.

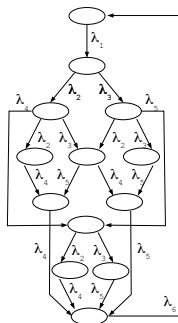
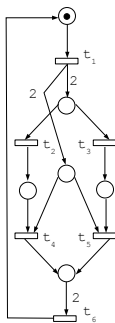
## First observations

- **The weights for choice policy are no more required** since equality of two samples has a null probability.  
(*due to continuity of distributions*)
- The residual delay  $d_j - d_i$  of transition  $t_j$  knowing that  $t_i$  has fired (i.e.  $d_i$  is the shortest delay) has the same distribution as the initial delay.  
**Thus the memory policy is irrelevant.**

# Markovian Net and Markov Chain

Key observation: given a marking  $m$  with  $T_m = t_1, \dots, t_k$

- The sojourn time in  $m$  is an exponential distribution with rate  $\sum_i \lambda_i$ .
- The probability that  $t_i$  is the next transition to fire is  $\frac{\lambda_i}{(\sum_j \lambda_j)}$ .
- **Thus the stochastic process is a CTMC whose states are markings and whose transitions are the transitions of the reachability graph.**



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Product-form Petri Nets

# Generalizing Distributions for Nets

Modelling delays with exponential distributions is **reasonable** when:

- Only mean value information is known about distributions.
- Exponential distributions (or combination of them) are enough to approximate the “real” distributions.

Modelling delays with exponential distributions is **not reasonable** when:

- The distribution of an event is known and is poorly approximable with exponential distributions:  
*a time-out of 10 time units*
- The delays of the events have different magnitude orders:  
*executing an instruction versus performing a database request*

In the last case, the 0-Dirac distribution is required.

# Generalized Markovian Stochastic Petri Net (GSPN)

Generalized Markovian Stochastic Petri Nets (GSPN) are nets whose:

- *timed transitions* have exponential distributions,
- and *immediate transitions* have 0-Dirac distributions.

Their analysis is based on Markovian Renewal Process,  
a generalization of Markov chains.

# Markovian Renewal Process

A Markovian Renewal Process (MRP) fulfills:

- a *relative* memoryless property

$$\begin{aligned} Pr(S_{n+1} = s_j, T_n < \tau \mid S_0 = s_{i_0}, \dots, S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, \dots, S_n = s_i) \\ = Pr(S_{n+1} = s_j, T_n < \tau \mid S_n = s_i) \equiv Q[i, j, \tau] \end{aligned}$$

- The embedded chain is defined by:  $P[i, j] = \lim_{\tau \rightarrow \infty} Q[i, j, \tau]$
- The sojourn time  $\text{Soj}$  has a distribution defined by:

$$Pr(\text{Soj}[i] < \tau) = \sum_j Q[i, j, \tau]$$

## Analysis of a MRP

- The steady-state distribution (if there exists)  $\pi$  is deduced from the steady-state distribution of the embedded chain  $\pi'$  by:

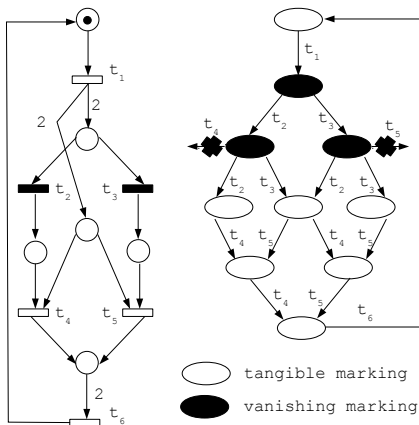
$$\pi(s_i) = \frac{\pi'(s_i)E(\text{Soj}[i])}{\sum_j \pi'(s_j)E(\text{Soj}[j])}$$

- Transient analysis is much harder ... but the reachability probabilities only depend on the embedded chain.

# A GSPN is a Markovian Renewal Process

## Observations

- Weights are required for immediate transitions.
- The *restricted* reachability graph corresponds to the embedded DTMC.



# Steady-State Analysis of a GSPN (1)

## Standard method for MRP

- Build the restricted reachability graph equivalent to the embedded DTMC
- Deduce the probability matrix  $P$
- Compute  $\pi^*$  the steady-state distribution of the visits of markings:  $\pi^* = \pi^* P$
- Compute  $\pi$  the steady-state distribution of the sojourn in tangible markings:

$$\pi(m) = \frac{\pi^*(m)\text{Soj}(m)}{\sum_{m' \text{ tangible}} \pi^*(m')\text{Soj}(m')}$$

How to eliminate the vanishing markings sooner in the computation?



# Steady-State Analysis of a GSPN (2)

## An alternative method

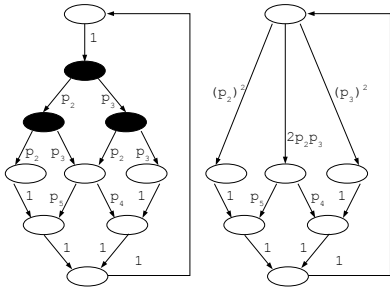
- As before, compute the transition probability matrix  $P$ .
- Compute the transition probability matrix  $P'$  between tangible markings.
- Compute  $\pi'^*$  the (relative) steady-state distribution of the visits of tangible markings:  $\pi'^* = \pi'^* P'$ .
- Compute  $\pi$  the steady-state distribution of the sojourn in tangible markings:

$$\pi(m) = \frac{\pi'^*(m)\text{Soj}(m)}{\sum_{m' \text{ tangible}} \pi'^*(m')\text{Soj}(m')}$$

## Computation of $P'$

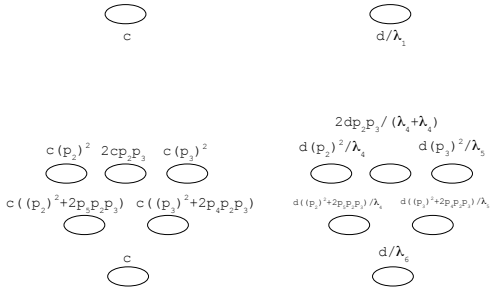
- Let  $P_{X,Y}$  the probability transition matrix from subset  $X$  to subset  $Y$ .
- Let  $V$  (resp.  $T$ ) be the set of vanishing (resp. tangible) markings.
- $P' = P_{T,T} + P_{T,V}(\sum_{n \in \mathbb{N}} P_{V,V}^n)P_{V,T} = P_{T,T} + P_{T,V}(Id - P_{V,V})^{-1}P_{V,T}$
- Iterative (resp. direct) computations uses the first (resp. second) expression.

# Steady-State Analysis: Illustration



$$p_2 = w_2 / (w_2 + w_3) \quad p_3 = w_3 / (w_2 + w_3)$$

$$p_4 = \lambda_4 / (\lambda_4 + \lambda_5) \quad p_5 = \lambda_5 / (\lambda_4 + \lambda_5)$$



"c" and "d" are normalizing constants

# Outline

Stochastic Petri Net

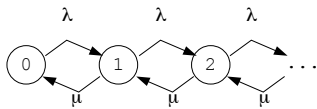
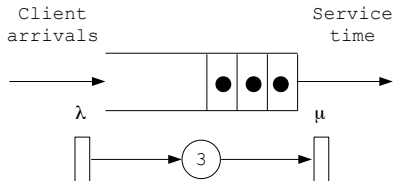
Markov Chain

Markovian Stochastic Petri Net

Generalized Markovian Stochastic Petri Net (GSPN)

5 Product-form Petri Nets

# Steady-State Analysis of a Queue



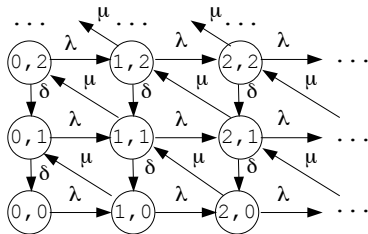
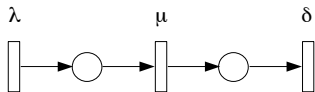
A (Markovian) queue is a CTMC

- Interarrival time: exponential distribution with parameter  $\lambda$
- Service time: exponential distribution with parameter  $\mu$

Let  $\rho = \frac{\lambda}{\mu}$  be the *utilization*

- The steady-state distribution  $\pi_\infty$  exists iff  $\rho < 1$
- The probability of  $n$  clients in the queue is  $\pi_\infty(n) = \rho^n(1 - \rho)$

# Analysis of Two Queues in Tandem

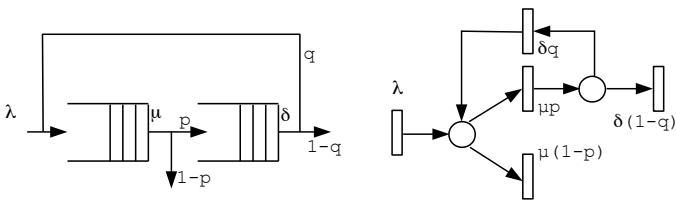


**Observation.** The associated Markov chain is more complex than the one corresponding to two isolated queues. However ...

Assume  $\rho_1 = \frac{\lambda}{\mu} < 1$  and  $\rho_2 = \frac{\lambda}{\delta} < 1$

- The steady-state distribution  $\pi_\infty$  exists.
- The probability of  $n_1$  clients in queue 1 and  $n_2$  clients in queue 2 is  $\pi_\infty(n_1, n_2) = \rho_1^{n_1} (1 - \rho_1) \rho_2^{n_2} (1 - \rho_2)$
- It is the **product** of the steady-state distributions corresponding to two isolated queues.

# Analysis of an Open Queuing Network



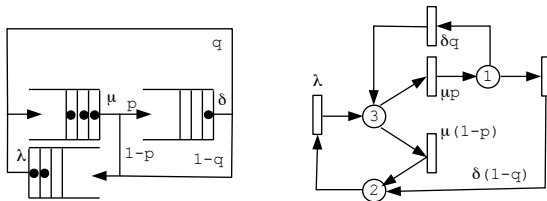
In a steady-state

- Define the (input and output) flow through queue 1 (resp. 2) as  $\gamma_1$  (resp.  $\gamma_2$ ).
- Then  $\gamma_1 = \lambda + q\gamma_2$  and  $\gamma_2 = p\gamma_1$ . Thus  $\gamma_1 = \frac{\lambda}{1-pq}$  and  $\gamma_2 = \frac{p\lambda}{1-pq}$

Assume  $\rho_1 = \frac{\gamma_1}{\mu} < 1$  and  $\rho_2 = \frac{\gamma_2}{\delta} < 1$

- The steady-state distribution  $\pi_\infty$  exists.
- The probability of  $n_1$  clients in queue 1 and  $n_2$  clients in queue 2 is  $\pi_\infty(n_1, n_2) = \rho_1^{n_1} (1 - \rho_1) \rho_2^{n_2} (1 - \rho_2)$
- It is the **product** of the steady-state distributions corresponding to two isolated queues.

# Analysis of a Closed Queuing Network



## Visit ratios (up to a constant)

- Define the visit ratio flow of queue  $i$  as  $v_i$ .
- Then  $v_1 = v_3 + qv_2$ ,  $v_2 = pv_1$  and  $v_3 = (1-p)v_1 + (1-q)v_2$ .  
Thus  $v_1 = 1$ ,  $v_2 = p$  and  $v_3 = 1 - pq$ .

Define  $\rho_1 = \frac{v_1}{\mu}$ ,  $\rho_2 = \frac{v_2}{\delta}$  and  $\rho_3 = \frac{v_3}{\lambda}$

- The steady-state probability of  $n_i$  clients in queue  $i$  is  $\pi_\infty(n_1, n_2, n_3) = \frac{1}{G} \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3}$  (with  $n_1 + n_2 + n_3 = n$ )
- where  $G$  the normalizing constant can be efficiently computed by dynamic programming.

# Queuing Networks and Petri Nets

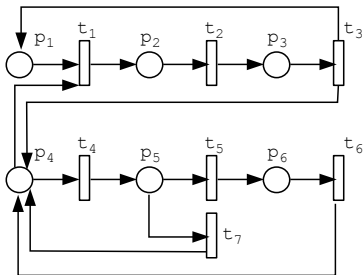
## Observations

- A (single client class) queuing network can easily be represented by a Petri net.
- Such a Petri net is a *state machine*: every transition has at most a single input and a single output place.

Can we define a more general subclass of Petri nets with a product form for the steady-state distribution?



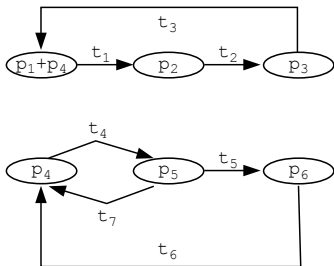
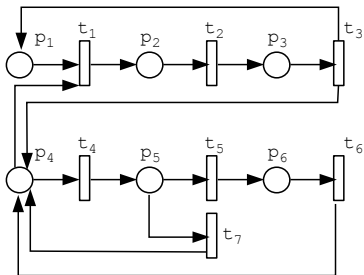
# Product Form Stochastic Petri Nets (PFSPN)



## Principles

- Transitions can be partitioned into subsets corresponding to several classes of clients with their specific activities
- Places model resources shared between the clients.
- Client states are implicitly represented.

# Bags and Transitions in PFSPN

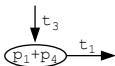
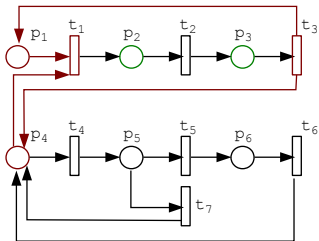


## The resource graph

- The vertices are the input and the output bags of the transitions.
- Every transition of the net  $t$  yields a graph transition  $\bullet t \xrightarrow{t} t \bullet$
- Client classes correspond to the connected components of the graph.

**First requirement: The connected components of the graph must be strongly connected.**

# Witnesses in PFSPN



**Vector  $-p_2-p_3$  is a witness for bag  $p_1+p_4$ :**

$$(-p_2-p_3) \cdot W(t_3)=1$$

$$(-p_2-p_3) \cdot W(t_1)=-1$$

$$(-p_2-p_3) \cdot W(t)=0 \text{ for every other } t$$

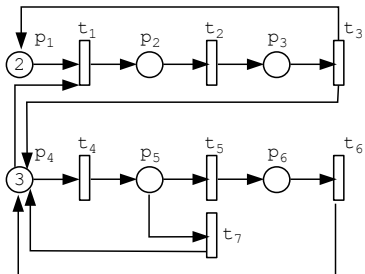
where  $W$  is the incidence matrix

## Witness for a bag $b$

- Let  $In(b)$  (resp.  $Out(b)$ ) the transitions with input (resp. output)  $b$ .
- Let  $v$  be a place vector,  $v$  is a *witness* for  $b$  if:
  - $\forall t \in In(b) v \cdot W(t) = -1$  (where  $W(t)$  is the incidence of  $t$ )
  - $\forall t \in Out(b) v \cdot W(t) = 1$
  - $\forall t \notin In(b) \cup Out(b) v \cdot W(t) = 0$

**Second requirement: Every bag must have a witness.**

# Steady-State Distributions of PFSPN



**The reachability space:**

$$m(p_1) + m(p_2) + m(p_3) = 2$$

$$m(p_4) + m(p_5) + m(p_6) = m(p_1) + 1$$

## Steady-state distribution

- Assume the requirements are fulfilled, with  $w(b)$  the witness for bag  $b$ .
- Compute the ratio visit of bags  $v(b)$  on the resource graph.
- The output rate of a bag  $b$  is  $\mu(b) = \sum_{t \bullet b} \mu(t)$  with  $\mu(t)$  the rate of  $t$ .
- Then:  $\pi_\infty(m) = \frac{1}{G} \prod_b \left( \frac{v(b)}{\mu(b)} \right)^{w(b) \cdot m}$

**Observation.** The normalizing constant can be efficiently computed if the reachability space is characterized by linear place invariants.

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