Homework 1 - Probabilistic aspects of computer science

The objective of this homework is to establish the existence and unicity of a steady-state distribution for an ergodic Markov chain *without using the fundamental theorem of renewal process.*

1 Renewal process

In order to handle the case of a defective distribution over $\mathbb{N}^*$, we consider that a distribution is defined by $\{f_i\}_{i\in\mathbb{N}^*\cup\infty}$. Its mean value μ is defined by: $\mu \stackrel{\text{def}}{=} \sum_{i\in\mathbb{N}\cup\infty} if_i$ and has been shown to be equal to:

$$\mu = \sum_{k\in\mathbb{N}} \rho_k \text{ where } \rho_k \stackrel{\text{def}}{=} \sum_{i>k} f_i \tag{1}$$

We recall an important renewal equation where u_i is the probability that i is a renewal instant:

$$\forall n \in \mathbb{N}\ \rho_0 u_n + \rho_1 u_{n-1} + \ldots + \rho_n u_0 = 1 \tag{2}$$

Let us define $\overline{u}_n \stackrel{\text{def}}{=} \frac{1}{n+1}\sum_{i\leq n} u_i$.

Question 1. Show that $\lim_{n\to\infty} \overline{u}_n = \frac{1}{\mu}$.

Hint: sum equation (2) from 0 *to* N *and then consider the cases where* μ *is infinite and finite.*

Let b be a non defective distribution over $\mathbb{N}$, and define v_i as the probability that i is a renewal instant in the delayed renewal process where b is the delay distribution and f is the renewal distribution.

$$\forall n \in \mathbb{N}\ v_n = b_0 u_n + b_1 u_{n-1} + \ldots + b_n u_0 \tag{3}$$

Let us define $\overline{v}_n \stackrel{\text{def}}{=} \frac{1}{n+1}\sum_{i\leq n} v_i$.

Question 2. Using equation (3), show that $\lim_{n\to\infty} \overline{v}_n = \frac{1}{\mu}$

2 Stationary distribution

Let $\mathcal{M}$ be a DTMC with initial state i. Define:

- $p_{ij}^{(k)}$ as the probability to be in state j at time k;
- $f_{ij}^{(k)}$ as the probability to be in state j at time k *for the first time excluding time 0*;
- $f_{ij}^{(\infty)}$ the probability to never reach j;
- $\mu_i \stackrel{\text{def}}{=} \sum_{k\in\mathbb{N}\cup\infty} k f_{ii}^{(k)}$ the mean return time to state i.

A state i is positive recurrent if and only if μ_i is finite.

In all the homework, $\mathcal{M}$ is irreducible: for all states i, j there exists k such that $p_{ij}^{(k)} > 0$.

Question 3. Show that in an irreducible DTMC, if i is recurrent then for any state j the probability to reach j from i is 1.

Observe that $p_{ji}^{(n)}$ corresponds to the probability that n is a renewal instant where the renewal is defined by meeting state i. Using question 1, a state i is positive recurrent if and only if $\lim_{n\to\infty} \overline{p}_{ii}^{(n)} > 0$ with $\overline{p}_{ii}^{(n)} \stackrel{\text{def}}{=} \frac{1}{n+1}\sum_{m\leq n} p_{ii}^{(m)}$.

Question 4. Using the previous characterization, prove that if some state i is positive recurrent, any state j is positive recurrent.

A distribution π over the states of $\mathcal{M}$ is said to be stationary if:

$$\pi = \pi\mathbf{P} \tag{4}$$

Question 5. Using equation (4), show that if π is a stationary distribution, for all state i and all $n \in \mathbb{N}$, $\pi[i] = \sum_{j\in S} \pi[j]\overline{p}_{ji}^{(n)}$.

Using questions 2 and 3, in a positive recurrent DTMC for all states i, j $\lim_{n\to\infty} \overline{p}_{ji}^{(n)} = \frac{1}{\mu_i}$.

Question 6. Using the previous characterization and question 5, show that if $\mathcal{M}$ is positive recurrent then there is at most one stationary distribution π which, when defined, is obtained by $\pi[i] = \frac{1}{\mu_i}$.

Given a positive recurrent DTMC $\mathcal{M}$, one defines another DTMC $\mathcal{M}^i$ where i is some state of $\mathcal{M}$. In words, $\mathcal{M}^i$ behaves as $\mathcal{M}$ except that states of $\mathcal{M}^i$ keep trace of the execution since the last visit of i (here we assume that $\mathcal{M}$ starts in i). More formally:

- $S^i = \{w \mid w \in i(S \setminus \{i\})^*\}$ i.e. words where i only occurs at the first position;
- for $k \neq i$ $\mathbf{P}^i[wj, wjk] = \mathbf{P}[j, k]$ and $\mathbf{P}^i[wj, i] = \mathbf{P}[j, i]$;
- All other transition probabilities are null.

Question 7. Using equation (1), show that:

$$\mu_i = \sum_{k=0}^{\infty} \sum_{w_1\ldots w_k \in (S\setminus\{i\})^*} p_{iw_1} \prod_{m=1}^{k-1} p_{w_m w_{m+1}}$$

Question 8. Show that the distribution π^i over S^i defined by:

$$\pi^i[i] \stackrel{\text{def}}{=} \frac{1}{\mu_i} \text{ and } \pi^i[iw_1 \ldots w_k] \stackrel{\text{def}}{=} \frac{p_{iw_1} \prod_{m=1}^{k-1} p_{w_m w_{m+1}}}{\mu_i}$$

is a stationary distribution of $\mathcal{M}^i$.

Question 9. Deduce that the distribution π over S defined by $\pi[j] \stackrel{\text{def}}{=} \sum_{wj\in S^i} \pi^i[j]$ is a stationary distribution of $\mathcal{M}$.

Question 10. Conclude that an irreducible DTMC has a (unique) stationary distribution if and only if it is positive recurrent.

3 Steady-state distribution

Given a DTMC $\mathcal{M}$, one defines the *square* DTMC $\mathcal{M}'$ with transition matrix $\mathbf{P}'$ over state space S^2 by:

$$\mathbf{P}'[(i, j), (i', j')] = \mathbf{P}[i, i']\mathbf{P}[j, j']$$

Question 11. Exhibit a finite irreducible DTMC such that its square Markov chain is not irreducible.

By definition, when $\mathcal{M}$ is irreducible and aperiodic, $gcd(n \mid f_{ii}^n > 0) = 1$. Thus using the arithmetic lemma of the lectures, there exists n_i such that for all $n \geq n_i$, $p_{ii}^n > 0$.

Question 12. Using the previous observation, show that given an irreducible and aperiodic DTMC, its square Markov chain is irreducible and aperiodic.

Question 13. Using questions 10 and 12, show that if $\mathcal{M}$ is ergodic then $\mathcal{M}'$ is ergodic.
Hint: Find an invariant distribution for $\mathcal{M}'$ using the one of $\mathcal{M}$.

Given a DTMC $\mathcal{M}$, one defines the *synchronized square* DTMC $\mathcal{M}''$ with transition matrix $\mathbf{P}''$ over state space S^2 by:

- For all $i \neq j, i', j'$ $\mathbf{P}''[(i,j),(i',j')] = \mathbf{P}[i,i']\mathbf{P}[j,j']$
- For all i, i', $\mathbf{P}''[(i,i),(i',i')] = \mathbf{P}[i,i']$
- All other transition probabilities are null.

In words, as long as the states of the pair are different, $\mathcal{M}''$ behaves as $\mathcal{M}'$. When they are identical, $\mathcal{M}''$ behaves as $\mathcal{M}$ maintaining the identity. Define (I_n, J_n) the random state of $\mathcal{M}''$ at time n.

Question 14. Let $i \in S$ and assume that $\mathcal{M}$ has an invariant distribution π. Define the initial distribution π'' of $\mathcal{M}''$ by:

$$\forall i \neq i', j \ \pi''[i,j] = \pi[j] \text{ and } \pi''[i',j] = 0$$

Prove that $|p_{ij}^{(n)} - \pi[j]| \leq \mathbf{Pr}(I_n \neq J_n)$.
Hint: consider $\mathbf{Pr}(I_n) = j$ *and* $\mathbf{Pr}(J_n) = j$.

Question 15. Show that $\mathbf{Pr}(I_n \neq J_n)$ is equal to the probability in $\mathcal{M}'$ with initial distribution π'' that the chain has not visited during time interval $[0,n]$ the subset of states $\{(s,s) \mid s \in S\}$. Using questions 3, 13 and 14 deduce that when $\mathcal{M}$ is ergodic, $\lim_{n\to\infty} |p_{ij}^{(n)} - \pi[j]| = 0$ (hence establishing that π is the steady-state distribution).