Alternative Analysis Methods for Stochastic Petri Nets

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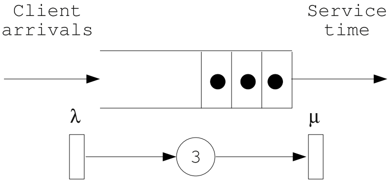
LSV CNRS & ENS Cachan & INRIA Saclay

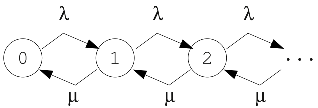
Advanced Course on Petri Nets, the 16th September 2010, Rostock

1. Product-form Petri Nets
2. Unbounded Petri Nets
3. Composition of Nets
4. Phase-Type Petri Nets

Plan

Steady-State Analysis of a Queue





A (Markovian) queue is a CTMC

- Interarrival time: exponential distribution with parameter λ
- Service time: exponential distribution with parameter μ

Let $\rho = \frac{\lambda}{\mu}$ be the *utilization*

- The steady-state distribution π_∞ exists iff $\rho < 1$
- The probability of n clients in the queue is $\pi_\infty(n) = \rho^n(1-\rho)$

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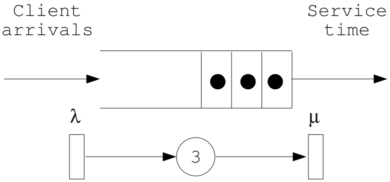

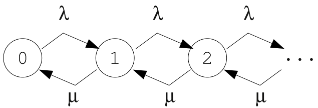

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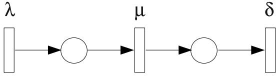
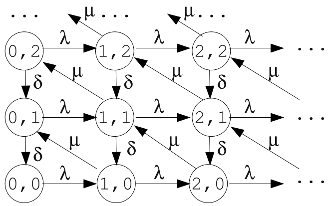
Analysis of Two Queues in Tandem





Observation. The associated Markov chain is more complex than the one corresponding to two isolated queues. However ...

Assume $\rho_1 = \frac{\lambda}{\mu} < 1$ and $\rho_2 = \frac{\lambda}{\delta} < 1$

- The steady-state distribution π_∞ exists.
- The probability of n_1 clients in queue 1 and n_2 clients in queue 2 is $\pi_\infty(n_1, n_2) = \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)$
- It is the product of the steady-state distributions corresponding to two isolated queues.

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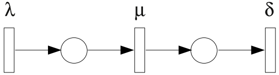

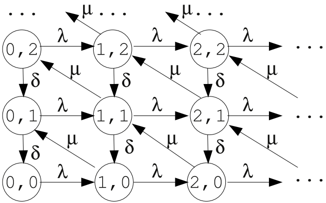

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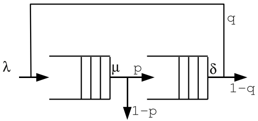

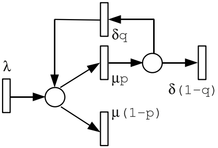

In a steady-state

- Define the (input and output) flow through queue 1 (resp. 2) as γ_1 (resp. γ_2).
- Then $\gamma_1 = \lambda + q\gamma_2$ and $\gamma_2 = p\gamma_1$. Thus $\gamma_1 = \frac{\lambda}{1-pq}$ and $\gamma_2 = \frac{p\lambda}{1-pq}$

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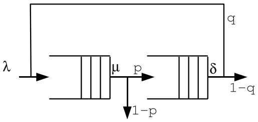

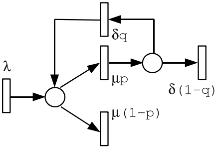

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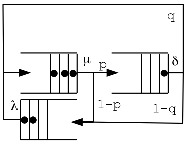

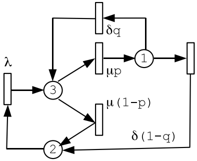

Visit ratios (up to a constant)

- Define the visit ratio flow of queue i as v_i.
- Then $v_1 = v_3 + qv_2$, $v_2 = pv_1$ and $v_3 = (1-p)v_1 + (1-q)v_2$. Thus $v_1 = 1$, $v_2 = p$ and $v_3 = 1 - pq$.

Define $\rho_1 = \frac{v_1}{\mu}$, $\rho_2 = \frac{v_2}{\delta}$ and $\rho_3 = \frac{v_3}{\lambda}$

- The steady-state probability of n_i clients in queue i is $\pi_\infty(n_1, n_2, n_3) = \frac{1}{G}\rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3}$ (with $n_1 + n_2 + n_3 = n$)
- where G the normalizing constant can be efficiently computed by dynamic programming.

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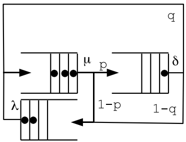

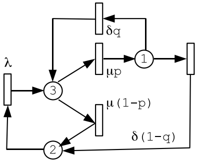

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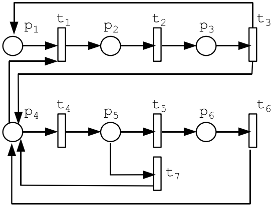
Queuing Networks and Petri Nets

Observations

- A (single client class) queuing network can easily be represented by a Petri net.
- Such a Petri net is a *state machine*: every transition has at most a single input and a single output place.

Can we define a more general subclass of Petri nets with a product form for the steady-state distribution?

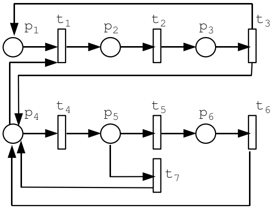
Product Form Stochastic Petri Nets (PFSPN)

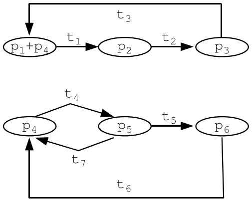


Principles

- Transitions can be partionned into subsets corresponding to several classes of clients with their specific activities
- Places model resources shared between the clients.
- Client states are implicitely represented.

Bags and Transitions in PFSPN



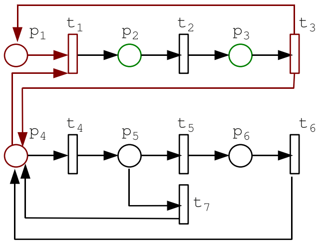


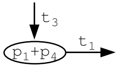
The resource graph

- The vertices are the input and the ouput bags of the transitions.
- Every transition of the net t yields a graph transition $^\bullet t \xrightarrow{t} t^\bullet$
- Client classes correspond to the connected components of the graph.

First requirement: The connected components of the graph must be strongly connected.

Witnesses in PFSPN





Vector $-p_2-p_3$ is a witness for bag p_1+p_4:

$(-p_2-p_3) \bullet W(t_3)=1$

$(-p_2-p_3) \bullet W(t_1)=-1$

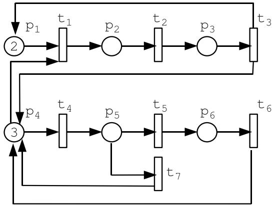
$(-p_2-p_3) \bullet W(t)=0$ for every other t

Witness for a bag b

- Let $In(b)$ (resp. $Out(b)$) the transitions with input (resp. output) b.
- Let v be a place vector, v is a *witness* for b if:
  - $\forall t \in In(b)\ v \cdot W(t) = -1$ (where $W(t)$ is the incidence of t)
  - $\forall t \in Out(b)\ v \cdot W(t) = 1$
  - $\forall t \notin In(b) \cup Out(b)\ v \cdot W(t) = 0$

Second requirement: Every bag must have a witness.

Steady-State Distributions of PFSPN



The reachability space:

$m(p_1) + m(p_2) + m(p_3) = 2$

$m(p_4) + m(p_5) + m(p_6) = m(p_1) + 1$

Steady-state distribution

- Assume the requirements are fulfilled, with $w(b)$ the witness for bag b.
- Compute the ratio visit of bags $v(b)$ on the resource graph.
- The output rate of a bag b is $\mu(b) = \sum_{t|^\bullet t=b} \mu(t)$ with $\mu(t)$ the rate of t.
- Then: $\pi_\infty(m) = \frac{1}{G} \prod_b \left(\frac{v(b)}{\mu(b)}\right)^{w(b)\cdot m}$

Observation. The normalizing constant can be efficiently computed if the reachability space is characterized by linear place invariants.

Some References

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Plan

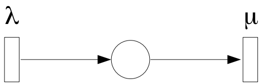
Product-form Petri Nets

2. Unbounded Petri Nets

Composition of Nets

Phase-Type Petri Nets

An Unbounded Petri Net



Q the infinitesimal generator

$$\begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & \dots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \dots \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \dots \\ \dots \end{pmatrix}$$

The steady-state distribution: $x_i = \left(\frac{\lambda}{\mu}\right)^i (1 - \frac{\lambda}{\mu})$

- Let x_i be the steady-state probability of i tokens in the place
- The steady-state equations are:
  1. $x_0\lambda - x_1\mu = 0$
  2. $\forall i \geq 1 \; x_{i-1}\lambda - x_i(\lambda + \mu) + x_{i+1}\mu = 0$
  3. $\sum_{i \in \mathbb{N}} x_i = 1$

Proof: Just check the equations! ... or use a simple trick.

An Alternative Steady-State Analysis

Assume there exists ρ such that:

- $\forall i \geq 1 \; x_i \geq x_{i-1}\rho$
- $\lambda - \rho(\lambda + \mu) + \rho^2\mu = 0$

Define $y_0 \equiv x_0$ and for all i $y_{i+1} \equiv y_i\rho$.

Then:

- $\forall i \; y_i \leq x_i$ and thus $sy \equiv \sum_i y_i \leq 1$
- $\forall i \geq 1 \; y_{i-1}\lambda - y_i(\lambda + \mu) + y_{i+1}\mu = 0$

Normalize y: for all i $z_i \equiv \frac{y_i}{sy}$

Then z fulfills the steady-state equations (except possibly the first one). So for all i $z_i = x_i$ (and in fact $y = z = x$).

Such a ρ exists : $\rho \equiv \frac{\lambda}{\mu}$

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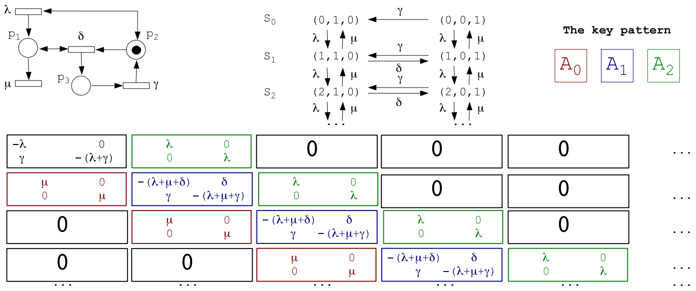

The steady-state distribution

Let X_i be the steady-state probability vector of markings with i tokens in p_1.
The steady-state equations are:

- $X_0 \cdot B + X_1 \cdot A_2 = 0$
- $\forall i \; X_i \cdot A_0 + X_{i+1} \cdot A_1 + X_{i+2} \cdot A_2 = 0$
- $\sum_{i \in \mathbb{N}} ||X_i|| = 1$

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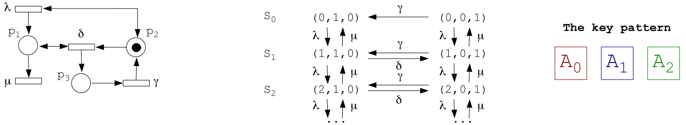

$\begin{matrix} -\lambda & 0 \\ \gamma & -(\lambda+\gamma) \end{matrix}$	$\begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix}$	0	0	0	...
$\begin{matrix} \mu & 0 \\ 0 & \mu \end{matrix}$	$\begin{matrix} -(\lambda+\mu+\delta) & \delta \\ \gamma & -(\lambda+\mu+\gamma) \end{matrix}$	$\begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix}$	0	0	...
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Nets with a Single Unbounded Place (1)

Let $\mathcal{N}$ be a net with a single unbounded place p

- Assume (for simplicity) that arcs around p are weighted by 1.
- The previous equation scheme holds for the steady-state distribution.

Let us try to mimic the previous analysis.
Assume there exists a non negative matrix R such that:
$\forall i \geq 1 \; X_{i+1} \geq X_i \cdot R$ and $A_0 + R \cdot A_1 + R^2 \cdot A_2 = 0$

Define Y by:

1. Solving $Y_0 \cdot B + Y_1 \cdot A_2 = 0 \wedge Y_0 \cdot A_0 + Y_1 \cdot (A_1 + R \cdot A_2) = 0$ omitting an arbitrary equation *(up to a constant)*
2. Inductively let for all $i \geq 1 \; Y_{i+1} \equiv Y_i \cdot R$

Then Y fulfils all the equations (except possibly one)

$$\forall i \geq 1 \; Y_i \cdot A_0 + Y_{i+1} \cdot A_1 + Y_{i+2} \cdot A_2 = Y_i \cdot (A_0 + R \cdot A_1 + R^2 \cdot A_2) = 0$$

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$X_1 \cdot \left(\sum_{i \in N} R^i\right) \leq \sum_{i \geq 1} X_i$ thus $\sum_{i \in N} R^i$ is finite implying the finiteness of $sy \equiv \sum_i ||Y_i||$.

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Nets with a Single Unbounded Place (2)

However Y can be normalized:

$X_1 \cdot \left(\sum_{i \in N} R^i\right) \leq \sum_{i \geq 1} X_i$ thus $\sum_{i \in N} R^i$ is finite
implying the finiteness of $sy \equiv \sum_i ||Y_i||$.

Normalize Y: for all i, $Z_i \equiv \frac{Y_i}{sy}$

Then Z fulfills the steady-state equations. So for all i, $Z_i = X_i$.

How to find R?

Looking for R: a fixed point approach

Observation. A_1 is invertible and $-A_1^{-1}$ is non negative.

Define $R_0 \equiv 0$ and $\forall n\ R_{n+1} \equiv -(A_0 + R_n^2 \cdot A_2)A_1^{-1}$.

Then by induction:

- R_n is a increasing sequence of non negative matrices
- $\forall i \geq 1\ X_{i+1} \geq X_i \cdot R_n$.

Thus the sequence R_n is bounded and so convergent.

Let $R \equiv \lim_{n\to\infty} R_n$.

Then: $\forall i \geq 2\ X_i \geq X_{i-1} \cdot R$ and $R = -(A_0 + R^2 \cdot A_2)A_1^{-1}$

Once R is computed, it remains to solve the following system:

- $X_0 \cdot B + X_1 \cdot A_2 = 0$, $X_0 \cdot A_0 + X_1 \cdot (A_1 + R \cdot A_2) = 0$
- $\sum_i X_i \cdot \mathbf{1} = 1$ *equivalent to*
  $||X_0|| + (X_1 \cdot \sum_{n\in\mathbb{N}} R^n)\mathbf{1} = 1$ *equivalent to*
  $||X_0|| + ||X_1 \cdot (Id - R)^{-1}|| = 1$

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Plan

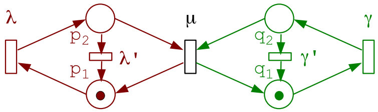
Product-form Petri Nets

Unbounded Petri Nets

3 Composition of Nets

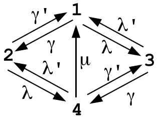
Phase-Type Petri Nets

A Composition of Two Nets



Reachable markings

1: p_1+q_1
2: p_1+q_2
3: p_2+q_1
4: p_2+q_2



The infinitesimal generator Q

$$\begin{pmatrix} -(\gamma+\lambda) & \gamma & \lambda & 0 \\ \gamma' & -(\gamma'+\lambda) & 0 & \lambda \\ \lambda' & 0 & -(\gamma+\lambda') & \gamma \\ \mu & \lambda' & \gamma' & -(\gamma'+\lambda'+\mu) \end{pmatrix}$$

Decomposition of the Generator

$$Q = \begin{pmatrix} -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & \lambda \\ \lambda' & 0 & -\lambda' & 0 \\ 0 & \lambda' & 0 & -\lambda' \end{pmatrix} + \begin{pmatrix} -\gamma & \gamma & 0 & 0 \\ \gamma' & -\gamma' & 0 & 0 \\ 0 & 0 & -\gamma & \gamma \\ 0 & 0 & \gamma' & -\gamma' \end{pmatrix} + \mu \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Decomposition w.r.t. the activities inside the net

- A matrix for the local activity of every component Q_i *(here two components)*
- A matrix for every synchronised transition B_t with coefficients in $\{-1, 0, 1\}$ *(here a single transition)*

How to exploit the regularity of these matrices?

Decomposition of the Generator

$$Q = \begin{pmatrix} -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & \lambda \\ \lambda' & 0 & -\lambda' & 0 \\ 0 & \lambda' & 0 & -\lambda' \end{pmatrix} + \begin{pmatrix} -\gamma & \gamma & 0 & 0 \\ \gamma' & -\gamma' & 0 & 0 \\ 0 & 0 & -\gamma & \gamma \\ 0 & 0 & \gamma' & -\gamma' \end{pmatrix} + \mu \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

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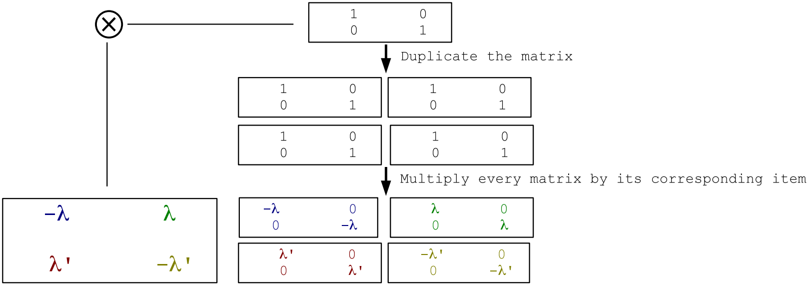
A Key Operation: the Tensorial Product (1)

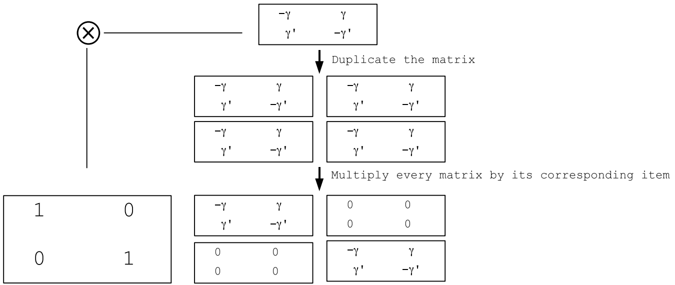


Q_1, the matrix associated with the first component fulfils:

$$Q_1 = Ql_1 \otimes Id$$

- where Ql_1 is the generator of the local Markov chain of the first component;
- where Id the identity matrix witnesses the independence from the second component.

A Key Operation: the Tensorial Product (2)

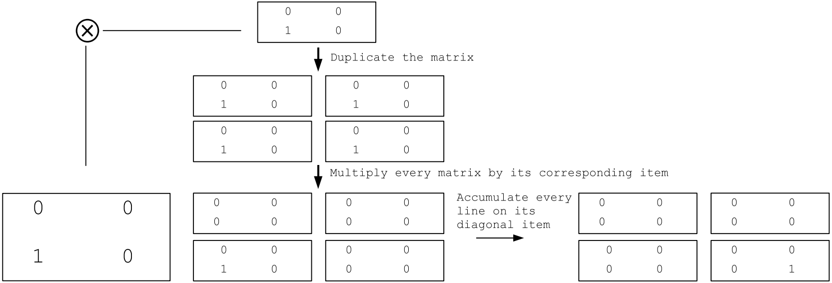


Q_2, the matrix associated with the first component fulfils:

$$Q_2 = Id \otimes Ql_2$$

- where Ql_2 is the generator of the local Markov chain of the second component;
- where Id the identity matrix witnesses the independence from the first component.

A Key Operation: the Tensorial Product (3)



B_t, the matrix associated with the synchronized transition fulfils:

$$B_t = B_{t,1} \otimes B_{t,2} - \mathbf{D}(B_{t,1} \otimes B_{t,2})$$

- where $B_{t,i}$ is the indicator of local state change due to t in the ith component;
- where $\mathbf{D}$ is the matrix operator summing the items of a line in the diagonal item.

Steady-State Analysis via Tensorial Decomposition

Iterative computation of the steady-state distribution

- Select any initial distribution π_0.
- Iterate $\pi_{n+1} = \pi_n(Id + \frac{1}{c} \cdot Q)$
  with c any value greater than $\max_i(|Q[i,i]|)$
- Stop when the successive values are enough close

Observation

The iterative approach is based on the product of a vector by a matrix.

When $C = A \otimes B$ computing $v \cdot C$ can be done:

.

- without computing C thus saving space;
- and more efficiently thus saving time.

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The Vector-Matrix Multiplication

$$\begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{11} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{11} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (x_{11}a_{11} + x_{21}a_{21})b_{11} + (x_{12}a_{11} + x_{22}a_{21})b_{21} \\ (x_{11}a_{11} + x_{21}a_{21})b_{12} + (x_{12}a_{11} + x_{22}a_{21})b_{22} \\ (x_{11}a_{12} + x_{21}a_{22})b_{11} + (x_{12}a_{12} + x_{22}a_{22})b_{21} \\ (x_{11}a_{12} + x_{21}a_{22})b_{12} + (x_{12}a_{12} + x_{22}a_{22})b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} z_{11}b_{11} + z_{12}b_{21} \\ z_{11}b_{12} + z_{12}b_{22} \\ z_{21}b_{11} + z_{22}b_{21} \\ z_{21}b_{12} + z_{22}b_{22} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} z_{11} \\ z_{12} \end{pmatrix} \cdot B \\ \begin{pmatrix} z_{21} \\ z_{22} \end{pmatrix} \cdot B \end{pmatrix}$$

where $\begin{pmatrix} z_{11} & z_{21} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} \end{pmatrix} \cdot A$ and $\begin{pmatrix} z_{12} & z_{22} \end{pmatrix} = \begin{pmatrix} x_{12} & x_{22} \end{pmatrix} \cdot A$

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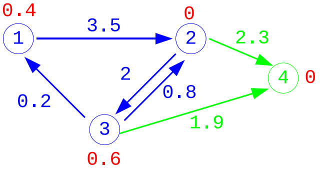
Plan

Phase-Type Distribution

A phase-type distribution is defined by:

- A continuous-time Markov chain with a single terminal scc consisting in an absorbing state.
- The associated distribution is the time to reach the absorbing state.



Interest of phase-type distributions

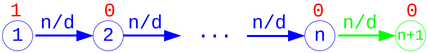
- Any distribution can be approximated as close as possible by a phase-type distribution.
- Warning: in some cases the number of states to obtain a "good" approximation can be prohibitive.

Approximating a Dirac Distribution

A random variable X with d-Dirac distribution is defined by $\mathbf{Pr}(X = d) = 1$

Erlang distributions

- A sequence of n non absorbing states with output rate n/d ending in the absorbing state.
- The mean value of the absorbing time is d.
- The variance of the absorbing time is $\frac{d^2}{n}$; so it goes to 0 when n goes to ∞. *(the coefficient of variation is* $\frac{1}{\sqrt{n}}$*)*



Erlang distributions are the appropriate approximations of Dirac distributions

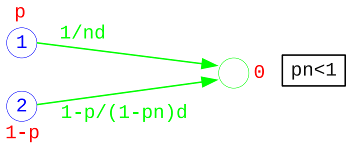
Increasing the coefficient of variation

Let d be the mean of some distribution with a great coefficient of variation.

Hyperexponential distributions

- A probabilistic choice between exponential distributions with different rates.
- The mean value of the absorbing time is the weighted average of mean values of these distributions.
- The coefficient of variation is always greater than 1.

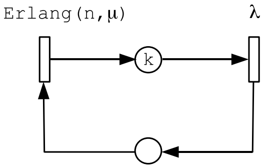
For instance, the given distribution can be approximated by a three-state hyperexponential distribution (with two parameters p, n).

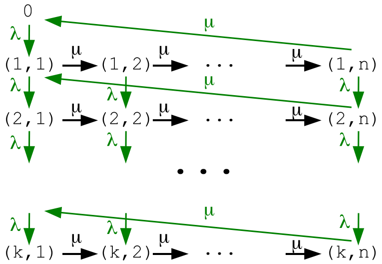


A Net with a Phase-Type Distribution

A net with phase-type distributions

- Its stochastic process is still a Markov chain.
- whose transitions are either external transitions modifying the current marking
- or internal transitions updating the stage of a phase-type distribution.





How to exploit the regularity of the chain?

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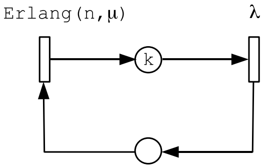

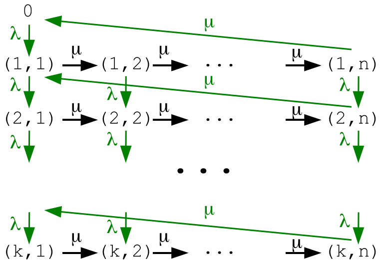

How to exploit the regularity of the chain?

Partition of the State Space

Partition the markings

- Markings with same enabled phase-type transitions are grouped.
- Let $\mathcal{M}$ be such a class of markings with $t_1, \ldots, t_d$ these enabled transitions.
- Then the corresponding states say $\mathcal{S}_{\mathcal{M}}$ in the Markov chain are $(m, q_1, \ldots, q_d)$ with $m \in \mathcal{M}$ and q_i being a non absorbing state of the distribution of t_i.

Structure of the generator between two classes $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}_{\mathcal{M}'}$

- Consider the block of Q indexed by $\mathcal{S}_{\mathcal{M}} \times \mathcal{S}_{\mathcal{M}'}$.
- Then a tensorial expression of.this block can be obtained where the involved matrices depends either on:
  1. either on the change of marking by exponential transitions or phase-type transitions when reaching absorbing states *(external transitions)*
  2. or on phase-type transitions that do no reach absorbing state *(internal transitions)*.

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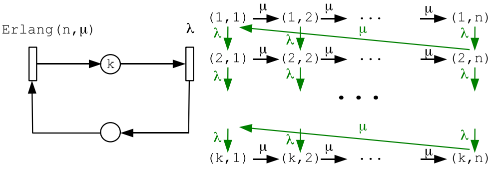
Example of Block Tensorial Expression



$$Q' = E \otimes Id + Id \otimes L + \mu(F \otimes G - \mathbf{D}(F \otimes G))$$

where E corresponds to the firing of the exponential transition,
L corresponds to the internal change of the distribution

$$E = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 \\ 0 & -\lambda & \lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & -\lambda & \lambda \\ 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} L = \begin{pmatrix} -\mu & \mu & 0 & 0 & \dots & 0 \\ 0 & -\mu & \mu & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & -\mu & \mu \\ 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

and F, G correspond to the firing of the phase-type transition

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix} G = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & 0 & 0 & 0 & 0 \\ 1 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

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