Probabilistic Aspects of Computer Science: CTMC

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MPRI M1

1. Mathematical Background
2. Renewal Processes with Non Arithmetic Distribution
3. Continuous Time Markov Chains (CTMC)
4. Finite CTMC

Plan

1. Mathematical Background

Renewal Processes with Non Arithmetic Distribution

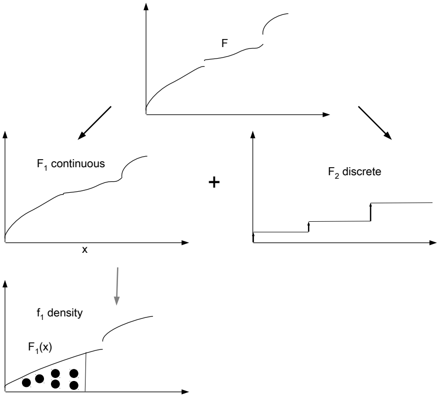
Continuous Time Markov Chains (CTMC**)**

Finite CTMC

Integration background

Using Caratheodory extension theorem, a *locally finite measure* μ_F on $\mathbb{R}^+$ is uniquely defined by a function F such that:

- F is non negative, non decreasing, right-continuous;
- $F(x) \stackrel{\text{def}}{=} \mu_F([0, x])$.

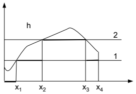


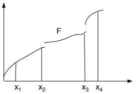
There are other kinds of measures like singular distributions (see Cantor distribution).

Integration background

The integral of a non negative measurable function h w.r.t. F is defined by:

$$\int_0^{\infty} h(x)F\{dx\} \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{k\in\mathbb{N}} \frac{k}{n}\mu_F\left(h^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right[\right)\right)$$





Riemann: x-decomposition and framing

Lebesgue: y-decomposition and lower bounding

1-Approximation: $F(x_2) - F(x_1) + F(x_4) - F(x_3) + 2(F(x_3) - F(x_2))$

When F is defined by $F(x) \stackrel{\text{def}}{=} \int_0^x f(\tau)d\tau + \sum_{x_i \le x} m_i$ where:

- $x_1, x_2, \ldots$ is a a sequence of points with mass $m_1, m_2, \ldots$;
- f is a non negative (measurable) density function.

Then:

$$\int_0^{\infty} h(x)F\{dx\} = \int_0^{\infty} h(x)f(x)dx + \sum_{i\in\mathbb{N}} h(x_i)m_i$$

Probability background

When $\lim_{x\to\infty} F(x) = 1$ (resp. ≤ 1) F is called a (resp. *defective*) *distribution*.
X, a random variable on $\mathbb{R}^+ \cup \{\infty\}$, has a distribution function F if:

$$\mathbf{Pr}(X \leq x) = F(x) \text{ (with } \mathbf{Pr}(X = \infty) = 1 - \lim_{x\to\infty} F(x) \text{)}$$

Let h be a non negative measurable function.
The expectation of $h(X)$ is defined by:

$$\mathbf{E}(h(X)) \stackrel{\text{def}}{=} \int_0^\infty h(x)F\{dx\} = \lim_{n\to\infty} \sum_{k\in\mathbb{N}} \frac{k}{n}\mathbf{Pr}\left(h(X) \in \left[\frac{k}{n}, \frac{k+1}{n}\right[\right)$$

Let X (resp. Y) be a non negative random variable with distribution F (resp. G).
Assume X and Y are independent.
The distribution of $X + Y$ is defined by the convolution $F \star G$:

$$F \star G(x) = \int_0^x F(x-y)G\{dy\} = \int_0^x G(x-y)F\{dy\}$$

The exponential distribution

Let F be defined by: $F(\tau) = 1 - e^{-\lambda\tau}$

Then F is the *exponential distribution* with *rate* $\lambda > 0$.

The exponential distribution is *memoryless*.

Let X be a random variable with a λ-exponential distribution.

$$\mathbf{Pr}(X > \tau' \mid X > \tau) = \frac{\mathbf{Pr}(X > \tau')}{\mathbf{Pr}(X > \tau)} = \frac{e^{-\lambda\tau'}}{e^{-\lambda\tau}} = e^{-\lambda(\tau'-\tau)} = \mathbf{Pr}(X > \tau' - \tau)$$

The minimum of exponential distributions is an exponential distribution.

Let Y be independent from X with μ-exponential distribution.

$$\mathbf{Pr}(\min(X, Y) > \tau) = e^{-\lambda\tau}e^{-\mu\tau} = e^{-(\lambda+\mu)\tau}$$

The minimal variable is selected proportionally to its rate.

$$\mathbf{Pr}(X < Y) = \int_0^\infty \mathbf{Pr}(Y > \tau) F_X\{d\tau\} = \int_0^\infty e^{-\mu\tau}\lambda e^{-\lambda\tau} d\tau = \frac{\lambda}{\lambda+\mu}$$

Convoluting the exponential distribution

The n^{th} convolution of a distribution F is defined by:

$$F^{n\star} \stackrel{\text{def}}{=} F \star \cdots \star F \qquad (n \text{ times})$$

Let f_n (resp. F_n) be the density (resp. distribution) of the n^{th} convolution of the λ-exponential distribution. Then:

$$f_n(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} \text{ and } F_n(x) = 1 - e^{-\lambda x} \sum_{0 \le m < n} \frac{(\lambda x)^m}{m!}$$

Sketch of proof

Recall that: $f_1(x) = \lambda e^{-\lambda x}$.

$$f_{n+1}(x) = \int_0^x f_n(x-u) f_1(u) du = \int_0^x \lambda e^{-\lambda(x-u)} \frac{(\lambda(x-u))^{n-1}}{(n-1)!} \lambda e^{-\lambda u} du$$
$$= \lambda e^{-\lambda x} \int_0^x \lambda \frac{(\lambda(x-u))^{n-1}}{(n-1)!} du = \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!}$$

Deduce F_{n+1} by:

$$\frac{d}{dx}\left(1 - e^{-\lambda x} \sum_{0 \le m \le n} \frac{(\lambda x)^m}{m!}\right) =$$
$$e^{-\lambda x}(\lambda \sum_{0 \le m \le n} \frac{(\lambda x)^m}{m!} - \sum_{0 \le m \le n-1} \lambda \frac{(\lambda x)^m}{m!}) = f_{n+1}(x)$$

Plan

Mathematical Background

2. Renewal Processes with Non Arithmetic Distribution

Continuous Time Markov Chains (CTMC)

Finite CTMC

Renewal process: definition

A renewal process is a very simple case of DES.

- It has a single state.
- The time intervals between events are reals obtained by sampling i.i.d. (independent and identically distributed) random variables with distribution F such that $F(0) = 0$.
- *Renewal instants* are the instants corresponding to the occurrence of events.

F is *arithmetic* if there exists $\alpha \in \mathbb{R}^+$ such that
the probability mass is concentrated on the set $\{k\alpha \mid k \in \mathbb{N}\}$.

Measure of renewal instants

The n^{th} renewal instant distribution is F convoluted $n-1$ times with itself: $F^{n\star}$

By convention $F^{0\star}(0) = 1$ *(the Dirac distribution concentrated in 0)*

The measure associated with renewal instants is $U \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} F^{n\star}$.

For instance $U(b) - U(a)$ is the mean number of renewal instants in $]a, b]$.

U is (locally) finite. More precisely for all $h \geq 0$,

there exists C_h such that $U\{I\} \leq C_h$ for all intervals I with length h.

Rewards for renewal processes

Let $z(\tau)$ be a (non increasing) function specifying the value of a renewal instant that occurred τ time units before (e.g. $\mathbf{1}_{\tau \leq \delta}$, $e^{-\tau}$, etc.).

Let $Z(x)$ be the cumulated discounted (w.r.t. z) value of renewal instants that occur up to x.

Then Z fulfills a renewal equation.
(and it is the single solution bounded on bounded intervals)

$$Z(x) = z(x) + \int_0^x Z(x-y) F\{dy\} \tag{1}$$

Z can also be expressed by:

$$Z(x) = U \star z(x) \stackrel{\text{def}}{=} \int_0^x z(x-y) U\{dy\}$$

Renewal theorems

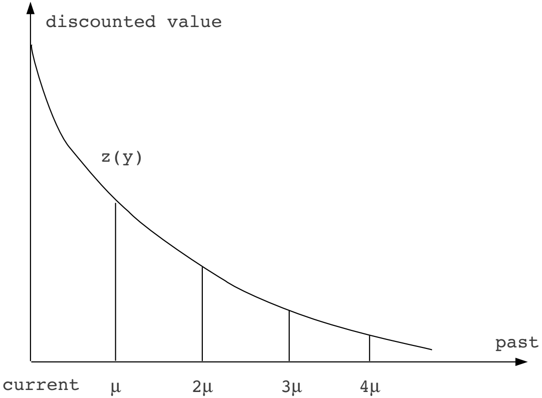
Let F be a non arithmetic distribution with (finite or infinite) expectation μ. Then:

$$\lim_{t \to \infty} U(t) - U(t-h) = \frac{h}{\mu} \text{ for all } h > 0$$

Let F be a non arithmetic distribution with (finite or infinite) expectation μ and z be a non increasing integrable function. Then:

$$\lim_{x \to \infty} Z(x) = \frac{1}{\mu} \int_0^\infty z(y)dy$$

Interpretation and informal justification



When x is large, there is approximatively one renewal instant uniformly distributed per interval $[x - i\mu, x - (i+1)\mu]$.

$$Z(x) \approx \frac{1}{\mu}\int_0^{\mu} z(y)dy + \frac{1}{\mu}\int_{\mu}^{2\mu} z(y)dy + \cdots$$

Generalizations

Instantaneous renewal process: $0 < p \stackrel{\text{def}}{=} F(0) < 1$

- Choose a number of renewals at the same instant with a Bernoulli law whose parameter is p.
- Choose the next renewal instant with distribution G defined by $G(0) \stackrel{\text{def}}{=} 0$ and $G(x) \stackrel{\text{def}}{=} (1-p)^{-1}(F(x)-p)$ for $x > 0$.

Let $V \stackrel{\text{def}}{=} \sum_{i \in \mathbb{N}} G^{i\star}$. Then $U = (1-p)^{-1}V$.

Delayed renewal process: The first renewal instant follows a distribution G. The mean number of renewal instants is given by $V \stackrel{\text{def}}{=} G \star U$. Then:

$$\boxed{\lim_{t \to \infty} V(t) - V(t-h) = \frac{h}{\mu} \text{ for all } h > 0}$$

Let z be a non increasing function such that $Z \stackrel{\text{def}}{=} U \star z$ is bounded. Then:

$$\boxed{\lim_{x \to \infty} (G \star Z)(x) = \frac{1}{\mu} \int_0^\infty z(y)dy}$$

Plan

Mathematical Background

Renewal Processes with Non Arithmetic Distribution

3. Continuous Time Markov Chains (CTMC)

Finite CTMC

Continous time Markov chains (CTMC)

A CTMC is a stochastic process which fulfills:

- The time interval between events T_n is a random variable whose distribution is the exponential one and whose rate only depends on state S_n.

$$\mathbf{Pr}(T_n \leq \tau \mid S_0 = s_{i_0}, ..., S_n = s_i, T_0 \leq \tau_0, ..., T_{n-1} \leq \tau_{n-1}) =$$

$$\mathbf{Pr}(T_n \leq \tau \mid S_n = s_i) \stackrel{\text{def}}{=} 1 - e^{-\lambda_i \cdot \tau}$$

- The selection of the state that follows the current state only depends on that state and the transition probabilities remain constant along the run:

$$\mathbf{Pr}(S_{n+1} = s_j \mid S_0 = s_{i_0}, ..., S_n = s_i, T_0 \leq \tau_0, ..., T_n \leq \tau_n) =$$

$$\mathbf{Pr}(S_{n+1} = s_j \mid S_n = s_i) \stackrel{\text{def}}{=} p_{ij} \stackrel{\text{def}}{=} \mathbf{P}[i, j]$$

$\mathbf{P}$ is called the *embedded* DTMC.

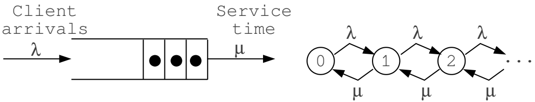
A CTMC is said *irreducible* if the embedded DTMC is irreducible.

Representation and illustration

Graph representation of a CTMC

- The set of vertices is the set of the states of the CTMC;
- There is an edge from s_i to s_j labelled by $\lambda_i p_{ij}$ if $p_{ij} > 0$ and $s_i \neq s_j$.

A single-server queue



For i>0

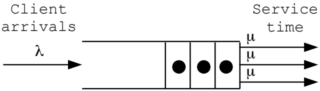
exit rate λ+μ

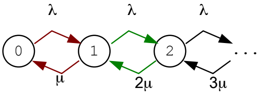
arrival probability λ/(λ+μ)

departure probability μ/(λ+μ)

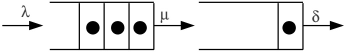
More illustrations

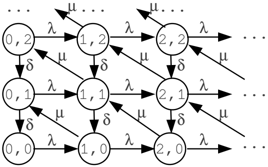
An infinite-server queue





A tandem queue





Transient behaviour

Let $\pi_{ij}(\tau) \stackrel{\text{def}}{=} \mathbf{Pr}(X(\tau) = j \mid X(0) = i)$ be the probability that the state is j at time τ knowing that at time 0 the state is i.

$\sum_{j \in S} \pi_{ij}(\tau) \leq 1$ but equality can be falsified.

$$0 \xrightarrow{1} 1 \xrightarrow{2} 2 \xrightarrow{4} 3 \xrightarrow{8} \cdots$$

$$\mathbf{E}\left(\sum_{n=0}^{\infty} T_n\right) = \sum_{n=0}^{\infty} \mathbf{E}(T_n) = 2$$

which implies $\mathbf{Pr}\left(\sum_{n=0}^{\infty} T_n \text{ is finite}\right) = 1$

which implies $\lim_{\tau \to \infty} \sum_{i \in S} \pi_{0i}(\tau) = 0$

Right-continuity of the transient distribution

By the memoryless property: $\pi_{ij}(\Delta + \tau) = \sum_k \pi_{ik}(\Delta)\pi_{kj}(\tau)$

- For all $i \neq j$, $\lim_{\tau \downarrow 0} \pi_{ii}(\tau) = 1$ and $\lim_{\tau \downarrow 0} \pi_{ij}(\tau) = 0$;
- For all i, j, π_{ij} is right-continuous and so measurable.

Sketch of proof

$\pi_{ii}(\tau) \geq e^{-\lambda_i \tau}$ implies $\lim_{\tau \downarrow 0} \pi_{ii}(\tau) = 1$

$\pi_{ii}(\tau) + \pi_{ij}(\tau) \leq 1$ implies $\lim_{\tau \downarrow 0} \pi_{ij}(\tau) = 0$

$\pi_{ij}(\tau + d\tau) = \sum_k \pi_{ik}(\tau)\pi_{kj}(d\tau)$ implies *(using dct)*
$\lim_{d\tau \downarrow 0} \pi_{ij}(\tau + d\tau) = \sum_k \pi_{ik}(\tau) \lim_{d\tau \downarrow 0} \pi_{kj}(d\tau) = \pi_{ij}(\tau)$

Backward differential equations

The *infinitesimal generator* of a CTMC, $\mathbf{Q}$, is defined by:

- $q_{ij} \stackrel{\text{def}}{=} \lambda_i \cdot p_{ij}$ for $i \neq j$
- $q_{ii} \stackrel{\text{def}}{=} (p_{ii} - 1)\lambda_i = -\sum_{j \neq i} q_{ij}$

The backward differential equation system

The family of functions $\{\pi_{ij}\}$ is differentiable and fulfills:

$$\frac{d\pi_{ij}(\tau)}{d\tau} = \sum_k q_{ik}\pi_{kj}(\tau)$$

Let matrix Π be defined by $\Pi[i,j] \stackrel{\text{def}}{=} \pi_{ij}$, the previous equation can be rewritten:

$$\frac{d\Pi}{d\tau} = \mathbf{Q} \cdot \Pi \qquad (2)$$

Proof of backward differential equations

A renewal equation based on the (possible) occurrence of the first event in $[0, \tau]$:

$$\pi_{ij}(\tau) = \mathbf{1}_{i=j} e^{-\lambda_i \tau} + \sum_k \int_0^\tau e^{-\lambda_i(\tau - x)} \lambda_i p_{ik} \pi_{kj}(x) dx$$

$$= e^{-\lambda_i \tau} \left(\mathbf{1}_{i=j} + \sum_k \lambda_i p_{ik} \int_0^\tau e^{\lambda_i x} \pi_{kj}(x) dx \right)$$

Every integral is a continuous function of τ.

The infinite sum of functions is normally convergent.

So the infinite sum is continuous implying the continuity of π_{ij}.

Every integral is differentiable and its derivative is equal to $e^{\lambda_i \tau} \pi_{kj}(\tau)$.

Due to the normal convergence of the sum of derivatives,

the infinite sum is differentiable implying the differentiability of π_{ij}.

Let us compute the derivative of π_{ij}:

$$\frac{d\pi_{ij}(\tau)}{d\tau} = e^{-\lambda_i \tau} \left(\sum_k \lambda_i p_{ik} e^{\lambda_i \tau} \pi_{kj}(\tau) \right) - \lambda_i \pi_{ij}(\tau) = \sum_k q_{ik} \pi_{kj}(\tau)$$

Forward differential equations

The forward differential equation system

Assume that $\sup(\lambda_i \mid i \in S)$ is finite. The family of functions $\{\pi_{ij}\}$ fulfills:

$$\frac{d\pi_{ij}(\tau)}{d\tau} = \sum_k \pi_{ik}(\tau) q_{kj}$$

which can be rewritten:

$$\frac{d\Pi}{d\tau} = \Pi \cdot \mathbf{Q} \tag{3}$$

Classification of states

As in a DTMC, a state of a CTMC may be:

- *transient*,
- *null recurrent*,
- or *positive recurrent* (equivalent to *ergodic*).

The transient and the recurrent characters only depend on the embedded DTMC.

In an irreducible CTMC, all states have same status.
(immediate for the transient character, proved later for positive/null recurrence)

Characterization of positive recurrence

Let i be a recurrent state and D_i be the mean time between two visits of i.

Then: $$\lim_{\tau\to\infty} \pi_{ii}(\tau) = \frac{1}{\lambda_i D_i}$$

Thus i is positive recurrent iff $\lim_{\tau\to\infty} \pi_{ii}(\tau) > 0$.

Sketch of proof

Let $z_i(\tau)$ be the (non increasing) probability to stay in i during at least τ time units: $z_i(\tau) \overset{\text{def}}{=} e^{-\lambda_i \tau}$.

$\pi_{ii}(\tau)$ fulfills the renewal equation.

$$\pi_{ii}(\tau) = z_i(\tau) + \int_0^\tau \pi_{ii}(\tau - y) F\{dy\}$$

where F is the (non arithmetic) distribution of the return time to i.

Using renewal theorem:

$$\lim_{\tau\to\infty} \pi_{ii}(\tau) = \frac{1}{D_i} \int_0^\infty e^{-\lambda_i \tau} d\tau = \frac{1}{\lambda_i D_i}$$

Positive recurrence and irreducible CTMC

Let i, j be two states of an irreducible recurrent CTMC.
Then i is positive recurrent iff j is positive recurrent.

Sketch of proof

Assume that i is positive recurrent.

There is a path from j to i and vice versa.

So given an arbitrary $\delta > 0$, $\pi_{ji}(\delta) > 0$ and $\pi_{ij}(\delta) > 0$.

Observe that: $\pi_{jj}(\tau + 2\delta) \geq \pi_{ji}(\delta)\pi_{ii}(\tau)\pi_{ij}(\delta)$.

Which implies: $\lim_{\tau\to\infty} \pi_{jj}(\tau + 2\delta) > 0$.

So j is positive recurrent.

Second characterization of positive recurrence

Let $\mathcal{C}$ be a *recurrent* irreducible CTMC. Then $\mathcal{C}$ is positive recurrent iff: there exists $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{Q} = 0$ with for all i, $\mathbf{u}_i > 0$ and $\sum_{i \in S} \mathbf{u}_i = 1$. In that case, $\mathbf{u}_i = \frac{1}{\lambda_i D_i}$

Preliminary observations

- Given states r, s, there exists $\rho > 0$ such that for all i, ${}_r\pi_{ri} = \rho \cdot {}_s\pi_{si}$
- $D_r = \sum_i \frac{{}_r\pi_{ri}}{\lambda_i}$ *(by linearity of expectations and monotone convergence theorem)*
- Let vectors $\mathbf{u}, \mathbf{v}$ fulfill for all i, $\mathbf{v}_i = \mathbf{u}_i \lambda_i$. Then $\mathbf{u} \cdot \mathbf{Q} = 0$ iff $\mathbf{v} \cdot \mathbf{P} = \mathbf{v}$

Sufficiency. Assume for all i, $\mathbf{u}_i > 0$, $\mathbf{u} \cdot \mathbf{Q} = 0$, and $\sum_{i \in S} \mathbf{u}_i = 1$
Then: $\mathbf{v}_i > 0, \mathbf{v} \cdot \mathbf{P} = \mathbf{v}$
So using recurrence of the embedded DTMC, $\exists \alpha > 0\ \forall i\ \mathbf{v}_i = \alpha \cdot {}_r\pi_{ri}$
$\sum_{i \in S} \mathbf{u}_i = \sum_{i \in S} \frac{\alpha\ {}_r\pi_{ri}}{\lambda_i} = \alpha D_r$ implying $D_r < \infty$.

Necessity. Assume for all i, $D_i < \infty$. Pick some state r.
$({}_r\pi_{ri})_{i \in S}$ and $({}_s\pi_{si})_{i \in S}$ are proportional. So $\frac{{}_r\pi_{ri}}{D_r} = \frac{{}_s\pi_{si}}{D_s}$ and $\frac{{}_r\pi_{ri}}{D_r} = \frac{{}_i\pi_{ii}}{D_i} = \frac{1}{D_i}$
Let $\mathbf{u}_i \stackrel{\text{def}}{=} \frac{1}{\lambda_i D_i} = \frac{{}_r\pi_{ri}}{\lambda_i D_r}$. So $\mathbf{u} \cdot \mathbf{Q} = 0$. Moreover $\sum_i \frac{1}{D_i \lambda_i} = \sum_i \frac{{}_r\pi_{ri}}{D_r \lambda_i} = 1$.

Third characterization of positive recurrence

Let $\mathcal{C}$ be an irreducible CTMC such that $\sup(\lambda_s \mid s \in S) < \infty$.
Then $\mathcal{C}$ is positive recurrent iff
there exists $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{Q} = 0$ with for all i, $\mathbf{u}_i > 0$ and $\sum_{i \in S} \mathbf{u}_i < \infty$.

Sketch of proof

The necessity is proved by the second characterization.

Let $\mathbf{u}$ be such that $\mathbf{u} \cdot \mathbf{Q} = 0$ for all i, $\mathbf{u}_i > 0$ and $\sum_{i \in S} \mathbf{u}_i$ is finite.

Let $\mathbf{v}_i \stackrel{\text{def}}{=} \mathbf{u}_i \lambda_i$.

Then $\mathbf{v} \cdot \mathbf{P} = \mathbf{v}$ and $\sum_{i \in S} \mathbf{v}_i \leq \sup_i(\lambda_i) \sum_{i \in S} \mathbf{u}_i$ is finite.

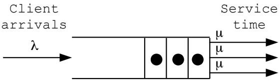
So the embedded DTMC is (positive) recurrent implying the recurrence of the $\mathcal{C}$.

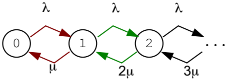
Applying the second characterization, $\mathcal{C}$ is positive recurrent.

Summary of the characterizations

Status	Characterization
Recurrent	The embedded DTMC is recurrent.
Positive Recurrent	(1) The embedded DTMC is recurrent *(implied by (2) when* $\sup(\lambda_i \mid i \in S) < \infty$*)* and (2) $\exists \mathbf{u} > 0\ \mathbf{u} \cdot \mathbf{Q} = 0 \wedge \sum_{i \in S} \mathbf{u}_i = 1$ *(*$\mathbf{u}$ *is the steady-state distribution)*

Analysis of an infinite-server queue





Recurrence versus Transience

$$x_1 = \frac{\lambda}{\mu+\lambda} x_2 \text{ and } \forall i \geq 2 \ x_i = \frac{\lambda}{i\mu+\lambda} x_{i+1} + \frac{i\mu}{i\mu+\lambda} x_{i-1}$$

It can be rewritten as:

$$x_1 = \frac{\lambda}{\mu+\lambda} x_2 \text{ and } \forall i \geq 2 \ x_{i+1} - x_i = \frac{i\mu}{\lambda}(x_i - x_{i-1})$$

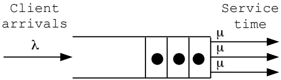
By induction:

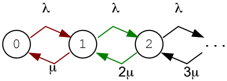
$$\forall i \geq 1 \ x_{i+1} - x_i > 0 \text{ and } \forall i \geq i_0 \stackrel{\text{def}}{=} \left\lceil \frac{\lambda}{\mu} \right\rceil \ x_{i+1} - x_i \geq x_i - x_{i-1}$$

Thus $\forall i \geq i_0 \ x_i \geq (i - i_0)(x_{i_0} - x_{i_0-1})$ implying that the x_i's are unbounded.

So the CTMC is recurrent.

Analysis of an infinite-server queue





Positive versus Null Recurrence

Global Balance Equation:

$$\lambda x_0 = \mu x_1 \text{ and } \forall i \geq 1 \; \lambda x_i + i\mu x_i = (i+1)\mu x_{i+1} + \lambda x_{i-1}$$

Local Balance Equation (by induction):

$$\forall i \geq 0 \; \lambda x_i = (i+1)\mu x_{i+1}$$

Let $\rho \stackrel{\text{def}}{=} \frac{\lambda}{\mu}$. For $i \geq 0$, $x_i = x_0 \frac{\rho^i}{i!}$

Thus the CTMC is positive recurrent and $\pi_\infty(i) = e^{-\rho} \frac{\rho^i}{i!}$

Plan

Mathematical Background

Renewal Processes with Non Arithmetic Distribution

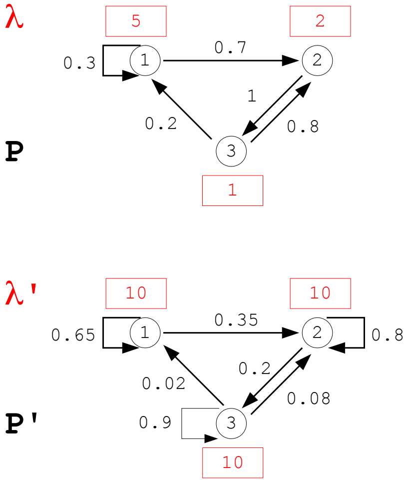
Continuous Time Markov Chains (CTMC)

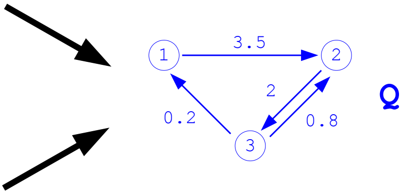
4 Finite CTMC

Uniformization

A CTMC

$$\lambda' \geq \sup(\lambda(i))$$
$$P'[i,j] = \lambda(i)P[i,j]/\lambda' \quad \text{for } i \neq j$$
$$P'[i,i] = 1 - \Sigma_{i\neq j} P'[i,j]$$





A *uniform version* of the CTMC *(equivalent w.r.t. the states)*

Transient analysis

Principle

- Construction of a uniform version of the CTMC $(\lambda, \mathbf{P})$
- Computation by case decomposition w.r.t. the number of transitions:
  $\pi(\tau) = \pi(0) \sum_{n \in \mathbb{N}} \mathbf{Pr}(n \text{ transitions in } [0, \tau]) \mathbf{P}^n = \pi(0) \sum_{n \in \mathbb{N}} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} \mathbf{P}^n$

Sketch of proof

Let F_n be the distribution of the n^{th} convolution of the λ-exponential distribution.

$$F_n(x) = 1 - e^{-\lambda x} \sum_{0 \leq m < n} \frac{(\lambda x)^m}{m!}$$

$$\boxed{\text{So } \mathbf{Pr}(n \text{ transitions in } [0, \tau]) = F_n(\tau) - F_{n+1}(\tau) = \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}}$$